

Around the tangent cone theorem

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Abstract A cornerstone of the theory of cohomology jump loci is the Tangent Cone theorem, which relates the behavior around the origin of the characteristic and resonance varieties of a space. We revisit this theorem, in both the algebraic setting provided by CDGA models, and in the topological setting provided by fundamental groups and cohomology rings. The general theory is illustrated with several classes of examples from geometry and topology: smooth quasi-projective varieties, complex hyperplane arrangements and their Milnor fibers, configuration spaces, and elliptic arrangements.

Key words: Algebraic model, cohomology ring, formality, resonance variety, characteristic variety, tangent cone, quasi-projective variety, configuration space, hyperplane arrangement, Milnor fiber, elliptic arrangement.

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1 Introduction

The Tangent Cone theorem relates two seemingly disparate sets of cohomology jump loci associated to a space X : the resonance varieties, which are constructed from information encoded in either the cohomology ring of X , or an algebraic model for this space, and the characteristic varieties, which depend on the *a priori* much more subtle information carried by the cohomology of X with coefficients in rank 1 local systems. We focus here on the interplay between these two sets of jump loci, which are even more tightly related under certain algebraic (positivity of weights), topological (formality), or geometric (quasi-projectivity) assumptions.

1.1 Resonance varieties

We start in §2 with a description of the various resonance varieties associated to a commutative, differential graded algebra (for short, a CDGA). We continue in §3 with the resonance varieties associated to a space X , using as input either its cohomology algebra or a suitable algebraic model, and discuss the algebraic version of the Tangent Cone theorem.

We will assume throughout that X is a reasonably nice space, to wit, a connected CW-complex with finitely many cells in each dimension. To such a space, we associate two types of resonance varieties. The classical ones are obtained from the cohomology algebra $A = H^*(X, \mathbb{C})$, by setting

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, \delta_a) \neq 0\}, \quad (1)$$

where, for each $a \in A^1$, we denote by (A, δ_a) the cochain complex with differentials $\delta_a: A^i \rightarrow A^{i+1}$ given by left-multiplication by a . Each set $\mathcal{R}^i(X)$ is a homogeneous subvariety of the complex affine space $A^1 = H^1(X, \mathbb{C})$.

Lately, an alternate definition of resonance has emerged (in works such as [22, 23, 24, 45, 58]), whereby one replaces the cohomology algebra by an algebraic model for X , that is, a commutative differential graded algebra (A, d) weakly equivalent to the Sullivan model of polynomial forms on X , as defined in [70]. We may then set up a cochain complex (A, δ_a) as above, but now with differentials given by $\delta_a(u) = au + du$, and define the resonance varieties $\mathcal{R}^i(A) \subset H^1(A)$ just as before.

Assuming now that each graded piece A^i is finite-dimensional, the sets $\mathcal{R}^i(A)$ are subvarieties of the affine space $H^1(A)$, which depend only on the isomorphism type of A . These varieties are not necessarily homogeneous; nevertheless, as shown in [45], the following inclusion holds,

$$\mathrm{TC}_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X), \quad (2)$$

where TC_0 denotes the tangent cone at $0 \in H^1(A)$.

Under some additional hypothesis, one can say more. Suppose our finite-type model (A, d) admits a \mathbb{Q} -structure compatible with that of the Sullivan model, and also has positive weights, in the sense of [70, 51]. Then, $\mathcal{R}^i(A)$ is a finite union of rationally defined linear subspaces of $H^1(A)$, and

$$\mathcal{R}^i(A) \subseteq \mathcal{R}^i(X). \quad (3)$$

1.2 Characteristic varieties

We turn in §4 to the characteristic varieties of a space X , and to the two types of tangent cones associated to them. This sets the stage for the topological version of the Tangent Cone theorem, which is treated in §5.

Unlike the resonance varieties, which arise from an algebraic model, the characteristic varieties arise from the chain complex of the universal abelian cover of the space. Let $\pi = \pi_1(X)$ be the fundamental group of X , let $\pi_{\mathrm{ab}} = H_1(X, \mathbb{Z})$ be its abelianization, and let $\mathrm{Char}(X) = \mathrm{Hom}(\pi_{\mathrm{ab}}, \mathbb{C}^*)$ be its group of complex-valued characters. Then

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H_i(X, \mathbb{C}_\rho) \neq 0\}, \quad (4)$$

where \mathbb{C}_ρ denotes the complex vector space \mathbb{C} , viewed as a module over the group algebra $\mathbb{C}[\pi_{\mathrm{ab}}]$ via $g \cdot z = \rho(g)z$, for $g \in \pi$ and $z \in \mathbb{C}$.

The relationship between the characteristic and resonance varieties of a space goes through the tangent cone construction. Let us start by identifying the tangent space at the identity to the complex algebraic group $\mathrm{Char}(X)$ with the complex affine space $H^1(X, \mathbb{C})$. Then, as shown in [41, 23], we have the following chain of inclusions:

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X), \quad (5)$$

where τ_1 denotes the ‘exponential tangent cone’ at the identity $1 \in \mathrm{Char}(X)$ (a finite union of rationally defined linear subspaces), and TC_1 denotes the usual tangent cone at 1 (a homogeneous subvariety).

The crucial property that bridges the gap between the two types of tangent cones to a characteristic variety and the corresponding resonance variety is that of formality, in the sense of Sullivan [70]. Given a 1-formal space, one of the main results from [23] establishes an isomorphism between the analytic germ of $\mathcal{V}^1(X)$ at 1 and the analytic germ of $\mathcal{R}^1(X)$ at 0.

More generally, if X has an algebraic model A with good finiteness properties, then, as shown in [22], the characteristic varieties $\mathcal{V}^i(X)$ may be identified around the identity with the resonance varieties $\mathcal{R}^i(A)$. Consequently, if X is formal (that is, the cohomology algebra of X , endowed with the zero differential, is weakly equivalent to the Sullivan model), then the following

‘Tangent Cone formula’ holds:

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X). \quad (6)$$

Consequently, if either one of the two inclusions in (5) fails to be an equality, the space X is not formal. Viewed this way, the Tangent Cone theorem can be thought of as a (quite powerful) formality obstruction.

1.3 Quasi-projective varieties

We conclude our overview of cohomology jump loci with an exploration of the Tangent Cone theorem in the framework of complex algebraic geometry. We start in §6 with the general theory of jump loci of smooth, quasi-projective varieties. We then specialize in §7 to complements of hyperplane arrangements and their Milnor fibers, and in §8 to complements of elliptic arrangements.

Let X be a smooth, complex quasi-projective variety. Work of Arapura [2], as recently sharpened by Budur and Wang [9], reveals a profound fact about the characteristic varieties $\mathcal{V}^i(X)$: they are all finite unions of torsion-translated subtori of the character group $\mathrm{Char}(X)$.

Every quasi-projective variety as above can be realized as the complement, $X = \overline{X} \setminus D$, of a normal-crossings divisor D in a smooth, complex projective variety \overline{X} . Given such a ‘good’ compactification, Morgan associates in [51] an algebraic model for our variety, $A(X) = A(\overline{X}, D)$. This ‘Gysin’ model is a finite-dimensional, rationally defined CDGA with positive weights, which is weakly equivalent to Sullivan’s model for X .

Using the aforementioned work of Arapura and Budur–Wang, as well as work of Dimca–Papadima [22], we obtain the following formulation of the Tangent Cone theorem for smooth, quasi-projective varieties X :

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A(X)) \subseteq \mathcal{R}^i(X). \quad (7)$$

In degree $i = 1$, the irreducible components of $\mathcal{V}^1(X)$ which pass through the identity are in one-to-one correspondence with the set \mathcal{E}_X of ‘admissible’ maps $f: X \rightarrow \Sigma$, where Σ is a smooth complex curve with $\chi(\Sigma) < 0$. This leads to a concrete description of the variety $\mathcal{R}^1(A(X))$, and of the variety $\mathcal{R}^1(X)$ when X is 1-formal.

Especially interesting is the case when $X = M(\mathcal{A})$ is the complement of an arrangement \mathcal{A} of hyperplanes in some complex vector space. The cohomology algebra $A = H^*(X, \mathbb{C})$ admits a combinatorial description, in terms of the intersection lattice of \mathcal{A} . Moreover, the CDGA $(A, 0)$ is a model for X ; thus, formula (7) holds with equalities throughout.

For an arrangement complement as above, work of Falk and Yuzvinsky [28] identifies the set \mathcal{E}_X with the set of multinets on sub-arrangements of \mathcal{A} , up to relabeling (see also [60]). This yields a completely combinatorial

description of the resonance variety $\mathcal{R}^1(X)$, and of the components of the characteristic variety $\mathcal{V}^1(X)$ passing through the identity.

Another smooth variety associated to an arrangement \mathcal{A} is the Milnor fiber $F = F(\mathcal{A})$, defined as the level set $Q = 1$, where Q is a defining polynomial for \mathcal{A} . The topology of this variety (even its first Betti number!) is much less understood. As shown by Zuber [74], though, the inclusion $\mathrm{TC}_1(\mathcal{V}^1(F)) \subset \mathcal{R}^1(F)$ can be strict; hence, F can be non-formal. Further understanding of how the Tangent Cone formula works in this context hinges on finding a good compactification for F , and then computing the corresponding Gysin model and its resonance varieties.

The machinery of cohomology jump loci can also be brought to bear in the study of elliptic arrangements. Let $E^{\times n}$ be the n -fold product of an elliptic curve E . An elliptic arrangement in $E^{\times n}$ is a finite collection of fibers of group homomorphisms $E^{\times n} \rightarrow E$. Assuming that all subspaces in the intersection poset of \mathcal{A} are connected, Bibby constructs in [7] a finite-dimensional, algebraic model for the complement, which can be thought of as a concrete version of the Gysin model.

A special case of this construction is the configuration space $\mathrm{Conf}(E, n)$ of n distinct, ordered points on E , itself a classifying space for the n -stranded pure braid group on the torus. We illustrate the general theory in a simple, yet instructive example. Direct computation shows that, for $X = \mathrm{Conf}(E, 3)$, the resonance variety $\mathcal{R}^1(A(X))$ is properly contained in $\mathcal{R}^1(X)$, thereby establishing the non-formality of X .

2 The resonance varieties of a cdga

We start with the resonance varieties associated to a commutative differential graded algebra, some of their properties, and various ways to compute them.

2.1 Commutative differential graded algebras

Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over the field \mathbb{C} . That is, $A = \bigoplus_{i \geq 0} A^i$ is a graded \mathbb{C} -vector space, endowed with a multiplication map $\cdot : A^i \otimes A^j \rightarrow A^{i+j}$ satisfying $u \cdot v = (-1)^{ij} v \cdot u$, and a differential $d : A^i \rightarrow A^{i+1}$ satisfying $d(u \cdot v) = du \cdot v + (-1)^i u \cdot dv$, for all $u \in A^i$ and $v \in A^j$.

Unless otherwise stated, we will assume throughout that A is connected, i.e., $A^0 = \mathbb{C}$, and of finite-type, i.e., A^i is finite-dimensional, for all $i \geq 0$.

Using only the underlying cochain complex structure of the CDGA, we let $Z^i(A) = \ker(d : A^i \rightarrow A^{i+1})$ and $B^i(A) = \mathrm{im}(d : A^{i-1} \rightarrow A^i)$, and set

$H^i(A) = Z^i(A)/B^i(A)$. The direct sum of the cohomology groups, $H^\bullet(A) = \bigoplus_{i \geq 0} H^i(A)$, inherits an algebra structure from A .

A morphism between two CDGAs, $\varphi: A \rightarrow B$, is both an algebra map and a cochain map. Consequently, φ induces a morphism $\varphi^*: H^\bullet(A) \rightarrow H^\bullet(B)$ between the respective cohomology algebras. We say that φ is a quasi-isomorphism if φ^* is an isomorphism. Likewise, we say φ is a q -isomorphism (for some $q \geq 1$) if φ^* is an isomorphism in degrees up to q and a monomorphism in degree $q+1$.

Two CDGAs A and B are *weakly equivalent* (or just *q -equivalent*) if there is a zig-zag of quasi-isomorphisms (or q -isomorphisms) connecting A to B , in which case we write $A \simeq B$ (or $A \simeq_q B$).

A CDGA (A, d) is said to be *formal* (or just *q -formal*) if it is weakly equivalent (or just q -equivalent) to its cohomology algebra, $H^\bullet(A)$, endowed with the zero differential.

Finally, we say that (A, d) is rationally defined if A is the complexification of a graded \mathbb{Q} -algebra $A_{\mathbb{Q}}$, and the differential d preserves $A_{\mathbb{Q}}$.

We will also consider the dual vector spaces $A_i = (A^i)^\vee := \text{Hom}(A^i, \mathbb{C})$, and the chain complex (A_\bullet, ∂) , where $\partial: A_{i+1} \rightarrow A_i$ is the dual to $d: A^i \rightarrow A^{i+1}$. If $H_i(A)$ are the homology groups of this chain complex, then, by the Universal Coefficients theorem, $H_i(A) \cong (H^i(A))^\vee$.

2.2 Resonance varieties

Our connectivity assumption on the CDGA (A, d) allows us to identify the vector space $H^1(A)$ with the cocycle space $Z^1(A)$. For each element $a \in Z^1(A) \cong H^1(A)$, we turn A into a cochain complex,

$$(A^\bullet, \delta_a): A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \dots, \quad (8)$$

with differentials given by $\delta_a^i(u) = a \cdot u + du$, for all $u \in A^i$. The cochain condition is verified as follows: $\delta_a^{i+1} \delta_a^i(u) = a^2 u + a \cdot du + da \cdot u - a \cdot du + ddu = 0$.

Computing the homology of these chain complexes for various values of the parameter a , and keeping track of the resulting Betti numbers singles out certain *resonance varieties* inside the affine space $H^1(A)$. More precisely, for each non-negative integer i , define

$$\mathcal{R}^i(A) = \{a \in H^1(A) \mid H^i(A^\bullet, \delta_a) \neq 0\}. \quad (9)$$

These sets can be defined for any connected CDGA. If A is of finite-type (as we always assume), the sets $\mathcal{R}^i(A)$ are, in fact, algebraic subsets of the ambient affine space $H^1(A)$. Clearly, $H^i(A^\bullet, \delta_0) = H^i(A)$; thus, the point $0 \in H^1(A)$ belongs to the variety $\mathcal{R}^i(A)$ if and only if $H^i(A) \neq 0$. Moreover, $\mathcal{R}^0(A) = \{0\}$.

When the differential of A is zero, the resonance varieties $\mathcal{R}^i(A)$ are homogeneous subsets of $H^1(A) = A^1$. In general, though, the resonance varieties of a CDGA are not homogeneous: see [45, Example 2.7] and Example 2.9 below.

The following lemma follows quickly from the definitions (see [45, Lemma 2.6] for details).

Lemma 2.1 ([45]). *Let $\varphi: A \rightarrow A'$ be a CDGA morphism, and assume φ is an isomorphism up to degree q , and a monomorphism in degree $q+1$, for some $q \geq 0$. Then the induced isomorphism in cohomology, $\varphi^*: H^1(A') \rightarrow H^1(A)$, identifies $\mathcal{R}^i(A)$ with $\mathcal{R}^i(A')$ for each $i \leq q$, and sends $\mathcal{R}^{q+1}(A)$ into $\mathcal{R}^{q+1}(A')$.*

Corollary 2.2. *If A and A' are isomorphic CDGAs, then their resonance varieties are ambiently isomorphic.*

The conclusions of Lemma 2.1 do not follow if we only assume that $\varphi: A \rightarrow A'$ is a q -isomorphism. This phenomenon is illustrated in [45, Example 2.7] and also in Example 2.9 below.

As shown in [56, 58], the resonance varieties behave reasonably well under tensor products:

$$\mathcal{R}^i(A \otimes A') \subseteq \bigcup_{p+q=i} \mathcal{R}^p(A) \times \mathcal{R}^q(A'). \quad (10)$$

Moreover, if the differentials of both A and A' are zero, then equality is achieved in the above product formula.

In a similar manner, we can define a homological version of resonance varieties, by considering the chain complexes (A_*, ∂^α) with differentials $\partial_i^\alpha = (\delta_a^i)^\vee$ for $\alpha \in H_1(A)$ dual to $a \in H^1(A)$, and setting

$$\mathcal{R}_i(A) = \{\alpha \in H_1(A) \mid H_i(A_*, \partial^\alpha) \neq 0\}. \quad (11)$$

Lemma 2.3. *For each $i \geq 0$, the duality isomorphism $H^1(A) \cong H_1(A)$ identifies the resonance varieties $\mathcal{R}^i(A)$ and $\mathcal{R}_i(A)$.*

Proof. By the Universal Coefficients theorem (over the field \mathbb{C}), we have that $H_i(A^*, \delta_a) \cong H_i(A_*, \partial^\alpha)$. The claim follows. \square

2.3 A generalized Koszul complex

Let us fix now a basis $\{e_1, \dots, e_n\}$ for the complex vector space $H^1(A)$, and let $\{x_1, \dots, x_n\}$ be the Kronecker dual basis for the vector space $H_1(A) = (H^1(A))^\vee$. In the sequel, we shall identify the symmetric algebra $\text{Sym}(H_1(A))$ with the polynomial ring $S = \mathbb{C}[x_1, \dots, x_n]$, and we shall view S as the coordinate ring of the affine space $H^1(A)$.

Consider now the cochain complex of free S -modules,

$$(A^\bullet \otimes S, \delta): \cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots, \quad (12)$$

where the differentials are the S -linear maps defined by

$$\delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j + du \otimes s \quad (13)$$

for all $u \in A^i$ and $s \in S$. As before, the fact that this is a cochain complex is easily verified. Indeed, $\delta^{i+1}\delta^i(u \otimes s)$ equals

$$\begin{aligned} & \sum_k e_k \left(\sum_j e_j u \otimes s x_j + du \otimes s \right) \otimes x_k + d \left(\sum_j e_j u \otimes s x_j + du \otimes s \right) \\ &= \sum_{j,k} e_k e_j u \otimes s x_j x_k + \sum_k e_k du \otimes s x_k - \sum_j e_j du \otimes s x_j \\ &= 0, \end{aligned}$$

where we used the fact that $e_k e_j = -e_j e_k$.

Remark 2.4. The cochain complex (12) is independent of the choice of basis $\{e_1, \dots, e_n\}$ for $H^1(A)$. Indeed, under the canonical identification $H^1(A) \otimes H_1(A) \cong \text{Hom}(H^1(A), H^1(A))$, the element $\sum_{j=1}^n e_j \otimes x_j$ used in defining the differentials δ^i corresponds to the identity map of $H^1(A)$. \diamond

Example 2.5. Let $E = \bigwedge(e_1, \dots, e_n)$ be the exterior algebra (with zero differential), and let $S = \mathbb{C}[x_1, \dots, x_n]$ be its Koszul dual. Then the cochain complex $(E^\bullet \otimes S, \delta)$ is simply the Koszul complex $K.(x_1, \dots, x_n)$. \diamond

More generally, if the CDGA A has zero differential, each boundary map $\delta^i: A^i \otimes S \rightarrow A^{i+1} \otimes S$ is given by a matrix whose entries are linear forms in the variables x_1, \dots, x_n . In general, though, the entries of δ^i may also have non-zero constant terms, as can be seen in Examples 2.9, 3.5, and 8.4 below.

The relationship between the cochain complexes (8) and (12) is given by the following lemma (for a more general statement, we refer to the proof of Lemma 8.8(1) from [22]).

Lemma 2.6. *The specialization of the cochain complex $A \otimes S$ at an element $a \in H^1(A)$ coincides with the cochain complex (A, δ_a) .*

Proof. Write $a = \sum_{j=1}^n a_j e_j \in H^1(A)$, and let $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$ be the maximal ideal at a . The evaluation map $\text{ev}_a: S \rightarrow S/\mathfrak{m}_a = \mathbb{C}$ is the ring morphism given by $g \mapsto g(a_1, \dots, a_n)$. The resulting cochain complex, $A(a) = A \otimes_S S/\mathfrak{m}_a$, has differentials $\delta^i(a)$ given by

$$\delta^i(a)(u) = \sum_{j=1}^n e_j u \otimes \text{ev}_a(x_j) + du = \sum_{j=1}^n e_j u \cdot a_j + du = a \cdot u + du. \quad (14)$$

Thus, $A(a) = (A, \delta_a)$, as claimed. \square

In a completely analogous fashion, we may define a chain complex

$$(A. \otimes S, \partial): \cdots \longrightarrow A_{i+1} \otimes S \xrightarrow{\partial_{i+1}} A_i \otimes S \xrightarrow{\partial_i} A_{i-1} \otimes S \longrightarrow \cdots \quad (15)$$

by essentially transposing the differentials of $(A^\bullet \otimes S, \delta)$. The previous lemma shows that the specialization of $(A. \otimes S, \partial)$ at an element $\alpha \in H_1(A)$ coincides with the chain complex $(A., \partial^\alpha)$.

2.4 Alternate views of resonance

As is well-known, the classical Koszul complex is exact. For an arbitrary CDGA, though, the cochain complex $(A^\bullet \otimes S, \delta)$ is not: its non-exactness is measured by the cohomology groups $H^i(A \otimes S)$, which are finitely generated modules over the polynomial ring S . This leads us to consider the support loci of these cohomology modules,

$$\tilde{\mathcal{R}}^i(A) = \text{supp}(H^i(A^\bullet \otimes S, \delta)), \quad (16)$$

viewed again as algebraic subsets of the affine space $H^1(A)$.

For instance, if $(E^\bullet \otimes S, \delta) = K.(x_1, \dots, x_n)$ is the Koszul complex from Example 2.5, the support loci $\tilde{\mathcal{R}}^i(E)$ vanish, for all $0 \leq i \leq n$.

We may also identify the polynomial ring S with the symmetric algebra on $H^1(A)$, viewed as the coordinate ring of $H_1(A)$. In this case, we have the support loci of the corresponding homology modules,

$$\tilde{\mathcal{R}}_i(A) = \text{supp}(H_i(A. \otimes S, \partial)), \quad (17)$$

which are algebraic subsets of the affine space $H_1(A)$. For a detailed discussion of support loci of chain complexes over an affine algebra, we refer to [57].

Since the ring S is no longer a field (or even a PID, unless $H_1(A) = 0$ or \mathbb{C}), the relation between these two types of support loci is not as straightforward as the one between the corresponding jump loci (see Example 2.9 below). Nevertheless, the cohomology jump loci and the homology support loci may be related in a filtered way, as follows.

Theorem 2.7. *For any finite-type CDGA (A, d) , and for any $q \geq 0$, the duality isomorphism $H^1(A) \cong H_1(A)$ restricts to an isomorphism*

$$\bigcup_{i \leq q} \mathcal{R}^i(A) \cong \bigcup_{i \leq q} \tilde{\mathcal{R}}_i(A).$$

Proof. As noted previously, the duality isomorphism $H^1(A) \cong H_1(A)$ identifies $\mathcal{R}^i(A)$ with $\mathcal{R}_i(A)$, for each $i \geq 0$.

On the other hand, we know that $(A. \otimes S, \partial)$ is a chain complex of free, finitely generated modules over the affine \mathbb{C} -algebra S . Therefore, by Theorem 2.5 from [57], we have that $\bigcup_{i \leq q} \mathcal{R}_i(A) = \bigcup_{i \leq q} \tilde{\mathcal{R}}_i(A)$, and the conclusion follows. \square

As noted previously, when the CDGA A has differential $d = 0$, the boundary maps δ and ∂ from the chain complexes (12) and (15) have entries which are linear forms in the variables of S . Consequently, the sets $\mathcal{R}^i(A)$ and $\tilde{\mathcal{R}}_i(A)$ are homogeneous subvarieties of the affine space $A^1 = H^1(A)$.

Corollary 2.8. *If A has zero differential, the resonance variety $\mathcal{R}^1(A) \subset A^1$ is the vanishing locus of the codimension 1 minors of the matrix of S -linear forms $\partial_2: A_2 \otimes S \rightarrow A_1 \otimes S$, or of its transpose, $\delta^1: A^1 \otimes S \rightarrow A^2 \otimes S$.*

Proof. Using Theorem 2.7 and the above discussion, we obtain the equality $\mathcal{R}^1(A) = \tilde{\mathcal{R}}_1(A)$. By definition, $\tilde{\mathcal{R}}_1(A)$ is the support locus of the S -module $H_1(A. \otimes S) = \ker \partial_1 / \text{im } \partial_2$. Writing $A_1 = \mathbb{C}^n$ and $S = \mathbb{C}[x_1, \dots, x_n]$, we have that $\partial_1 = (x_1 \cdots x_n)$. The conclusion follows. \square

We illustrate the theory with a simple, yet meaningful example, variants of which can also be found in [45, 24].

Example 2.9. Let A be the exterior algebra on generators a, b in degree 1, endowed with the differential given by $da = 0$ and $db = b \cdot a$. Then $H^1(A) = \mathbb{C}$, generated by a . Writing $S = \mathbb{C}[x]$, the chain complex (15) takes the form

$$A. \otimes S: S \xrightarrow{\partial_2 = \begin{pmatrix} 0 \\ x-1 \end{pmatrix}} S^2 \xrightarrow{\partial_1 = (x \ 0)} S. \quad (18)$$

Hence, $H_1(A. \otimes S) = S/(x-1)$, and so $\tilde{\mathcal{R}}_1(A) = \{1\}$. Using the above theorem, we conclude that $\mathcal{R}^1(A) = \{0, 1\}$.

Note that $\mathcal{R}^1(A)$ is a non-homogeneous subvariety of \mathbb{C} . Note also that $H^1(A. \otimes S) = S/(x)$, and so $\tilde{\mathcal{R}}^1(A) = \{0\}$, which differs from $\tilde{\mathcal{R}}_1(A)$.

Finally, let A' be the sub-CDGA generated by a . Clearly, the inclusion map, $\iota: A' \hookrightarrow A$, induces an isomorphism in cohomology. Nevertheless, $\mathcal{R}^1(A') = \{0\}$, and so the resonance varieties of A and A' differ, although A and A' are quasi-isomorphic. \diamond

Problem 2.10. Can the resonance varieties of a CDGA have positive-dimensional irreducible components not passing through 0?

3 The resonance varieties of a space

There are two basic types of resonance varieties that one can associate to a space, depending on which CDGA is used to approximate it. In this section, we discuss both types of resonance varieties, and several ways in which these varieties can be related.

3.1 The cohomology algebra

Throughout this section, X will be a connected, finite-type CW-complex. The first approach (which has been in use since the 1990s) is to take the cohomology algebra $H^*(X, \mathbb{C})$, endowed with the zero differential, and let $\mathcal{R}^i(X)$ be the resonance varieties of this CDGA. As indicated previously, these sets are homogeneous algebraic subvarieties of the affine space $H^1(X, \mathbb{C})$.

These varieties have been much studied in recent years, and have many practical applications, see e.g. [23, 56, 57, 59, 64, 65, 67] and the references therein. Let us just mention here two of their naturality properties.

First, the resonance varieties are homotopy-type invariants. More precisely, if $f: X \rightarrow Y$ is a homotopy equivalence, then the induced homomorphism $f^*: H^1(Y, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ restricts to an isomorphism $f^*: \mathcal{R}^i(Y) \xrightarrow{\cong} \mathcal{R}^i(X)$, for all $i \geq 0$, see e.g. [64].

Next, if $p: Y \rightarrow X$ is a finite, regular cover, then the induced homomorphism, $p^*: H^1(X, \mathbb{C}) \hookrightarrow H^1(Y, \mathbb{C})$, maps each resonance variety $\mathcal{R}^i(X)$ into $\mathcal{R}^i(Y)$, with equality if the group of deck transformations acts trivially on $H^*(Y, \mathbb{C})$, see e.g. [21, 67].

Yet the resonance varieties $\mathcal{R}^i(X)$ do not always provide accurate enough information about the space X , since the cohomology algebra may not be a (rational homotopy) model for X . It is thus important to look for alternate definitions of resonance in the non-formal setting.

3.2 The Sullivan model

The second approach is to use Sullivan's model of polynomial forms, $A_{\text{PL}}(X)$. This is a rationally defined CDGA, whose construction is inspired by the de Rham algebra of differential forms on a smooth manifold (see [70], [32]). In particular, the cohomology algebra $H^*(A_{\text{PL}}(X))$ is isomorphic as a graded algebra to $H^*(X, \mathbb{C})$, via an isomorphism preserving \mathbb{Q} -structures. For a finite simplicial complex K , the model $A_{\text{PL}}(K)$ admits a nice combinatorial description, closely related to the Stanley–Reisner ring of K (see [31]).

A connected, finite-type CW-complex X is said to be *formal* if its Sullivan model is formal, i.e., there is a weak equivalence $A_{\text{PL}}(X) \simeq (H^*(X, \mathbb{C}), 0)$

preserving \mathbb{Q} -structures. The notion of q -formality of a space is defined analogously. Of course, if X is formal, then it is q -formal, for all q . As a partial converse, if X is q -formal and $\dim X \leq q + 1$, then X is formal (see [44]).

Particularly interesting is the notion of 1-formality. It turns out that a space X as above is 1-formal if and only if its fundamental group, $\pi = \pi_1(X, x_0)$, is 1-formal, that is, if the Malcev–Lie algebra of π is the degree completion of a quadratic Lie algebra.

For instance, if $H^*(X, \mathbb{Q})$ is the quotient of a free CDGA by an ideal generated by a regular sequence, then X is a formal space (see [70]). In particular, if X has the rational cohomology of a torus, then X is formal. For more on these formality notions, we refer to [44, 54, 23, 55, 68].

When X is non-formal, the Sullivan model may have infinite-dimensional graded pieces. In particular, the sets $\mathcal{R}^i(A_{\text{PL}}(X))$ are not *a priori* algebraic sets. Thus, we will restrict our attention to spaces X for which $A_{\text{PL}}(X)$ can be replaced (up to weak equivalence) by a finite-type model (A, d) .

For this class of spaces, which includes many interesting examples of non-formal spaces, the resonance varieties $\mathcal{R}^i(A)$ may be viewed as algebraic subsets of the affine space $H^1(X, \mathbb{C}) \cong H^1(A)$.

3.3 An algebraic tangent cone theorem

Before proceeding, let us briefly recall a standard notion in algebraic geometry. Let $W \subset \mathbb{C}^n$ be a Zariski closed subset, defined by an ideal I in the polynomial ring $S = \mathbb{C}[z_1, \dots, z_n]$. The *tangent cone* of W at 0 is the algebraic subset $\text{TC}_0(W) \subset \mathbb{C}^n$ defined by the ideal $\text{in}(I) \subset S$ generated by the initial forms of all non-zero elements from I . This set is a homogeneous subvariety of \mathbb{C}^n , which depends only on the analytic germ of W at zero. In particular, $\text{TC}_0(W) \neq \emptyset$ if and only if $0 \in W$.

In the previous two subsections, we associated two types of resonance varieties to a space X having a finite-type model A . The next theorem, which may be viewed as an algebraic analogue of the Tangent Cone theorem, establishes a tight relationship between these two kinds of varieties.

Theorem 3.1 ([45]). *Let X be a finite-type CW-complex, and suppose there is a finite-type CDGA (A, d) such that $A_{\text{PL}}(X) \simeq A$. Then, for each $i \geq 0$, the tangent cone at 0 to the resonance variety $\mathcal{R}^i(A)$ is contained in $\mathcal{R}^i(X)$.*

As we shall see in Example 3.5 below, the inclusion $\text{TC}_0(\mathcal{R}^i(A)) \subseteq \mathcal{R}^i(X)$ may well be strict.

It seems natural to ask whether one can dispense in the above theorem with the hypothesis that the CDGA (A, d) be realized by a space X , and distill a purely algebraic statement from it.

Problem 3.2. Let (A, d) be a finite-type CDGA. For each $i \geq 0$, determine whether the tangent cone at 0 to $\mathcal{R}^i(A)$ is contained in $\mathcal{R}^i(H^*(A))$.

3.4 Positive weights

Under some additional hypothesis on the CDGA under consideration, one can say more about the nature of its resonance varieties.

Following Sullivan [70] and Morgan [51], we say that a rationally defined CDGA (A, d) has *positive weights* if each graded piece can be decomposed into weighted pieces, with positive weights in degree 1, and in a manner compatible with the CDGA structure. That is,

1. For each $i \geq 0$, there is a vector space decomposition, $A^i = \bigoplus_{\alpha \in \mathbb{Z}} A_\alpha^i$.
2. $A_\alpha^1 = 0$, for all $\alpha \leq 0$.
3. If $a \in A_\alpha^i$ and $b \in A_\beta^j$, then $ab \in A_{\alpha+\beta}^{i+j}$ and $da \in A_\alpha^{i+1}$.

A space X is said to have positive weights if its Sullivan model does. If X is formal, then X does have positive weights: simply set the weight of a cohomology class in $A = H^*(X, \mathbb{C})$ equal to its degree. On the other hand, as we shall see in §§6–8, the converse is far from true, even when X is a smooth, complex algebraic variety.

The existence of positive weights on a CDGA model A for X imposes stringent conditions on the resonance varieties of A , and leads to an even tighter relationship between the resonance varieties of the space and its model.

Theorem 3.3 ([22, 45]). *Let X be finite-type CW-complex, and suppose there is a rationally defined, finite-type CDGA (A, d) with positive weights, and a q -equivalence between $A_{\text{PL}}(X)$ and A preserving \mathbb{Q} -structures. Then, for each $i \leq q$,*

1. $\mathcal{R}^i(A)$ is a finite union of rationally defined linear subspaces of $H^1(A)$.
2. $\mathcal{R}^i(A) \subseteq \mathcal{R}^i(X)$.

Once again, it seems natural to ask whether one can dispense with the hypothesis that (A, d) be a model for a finite-type CW-complex X .

Problem 3.4. Let (A, d) be a finite-type CDGA with positive weights. For each $i \geq 0$, determine whether $\mathcal{R}^i(A)$ is contained in $\mathcal{R}^i(H^*(A))$, and whether $\mathcal{R}^i(A)$ is a finite union of rationally defined linear subspaces.

Example 3.5. Let X be the 3-dimensional Heisenberg nilmanifold, i.e., the circle bundle over the torus, with Euler number 1. Then $H^1(X, \mathbb{C}) = \mathbb{C}^2$, and all cup products of degree 1 classes vanish; thus, $\mathcal{R}^1(X) = H^1(X, \mathbb{C})$.

On the other hand, X admits as a model (A, d) the exterior algebra on generators a, b, c in degree 1, with differential $da = db = 0$ and $dc = a \wedge b$. Clearly, this is a finite-dimensional model, with positive weights: simply assign weight 1 to a and b , and weight 2 to c .

Writing $S = \mathbb{C}[x, y]$, the chain complex (15) takes the form

$$A. \otimes S: \dots \longrightarrow S^3 \xrightarrow{\begin{pmatrix} y & 0 & 0 \\ -x & 0 & 0 \\ 1 & -x & -y \end{pmatrix}} S^3 \xrightarrow{(x \ y \ 0)} S. \quad (19)$$

It follows that $H_1(A \otimes S) = S/(x, y)$, and so $\mathcal{R}^1(A) = \{0\}$, a proper subset of $\mathcal{R}^1(X) = \mathbb{C}^2$. \diamond

4 Characteristic varieties

We now turn to another type of homological jump loci associated to a space: the characteristic varieties, which keep track of jumps in the homology with coefficients in rank 1 local systems. Closely related objects are the support loci for the Alexander modules.

4.1 Homology jump loci for rank 1 local systems

As before, let X be a finite-type, connected CW-complex. Fix a base-point x_0 , and let $\pi = \pi_1(X, x_0)$ be its fundamental group. Finally, let $\text{Char}(X) = \text{Hom}(\pi, \mathbb{C}^*)$ be the algebraic group of complex-valued, multiplicative characters on π , with identity 1 corresponding to the trivial representation. The identity component of this group, $\text{Char}(X)^0$, is an algebraic torus of dimension $n = b_1(X)$; the other components are translates of this torus by characters corresponding to the torsion subgroup of $\pi_{\text{ab}} = H_1(X, \mathbb{Z})$.

For each character $\rho: \pi \rightarrow \mathbb{C}^*$, let \mathbb{C}_ρ be the corresponding rank 1 local system on X . The *characteristic varieties* of X are the jump loci for homology with coefficients in such local systems,

$$\mathcal{V}_i(X) = \{\rho \in \text{Char}(X) \mid H_i(X, \mathbb{C}_\rho) \neq 0\}. \quad (20)$$

In more detail, let $X^{\text{ab}} \rightarrow X$ be the maximal abelian cover, with group of deck transformations π_{ab} . Upon lifting the cell structure of X to this cover, we obtain a chain complex of $\mathbb{C}[\pi_{\text{ab}}]$ -modules,

$$\cdots \longrightarrow C_{i+1}(X^{\text{ab}}, \mathbb{C}) \xrightarrow{\partial_{i+1}^{\text{ab}}} C_i(X^{\text{ab}}, \mathbb{C}) \xrightarrow{\partial_i^{\text{ab}}} C_{i-1}(X^{\text{ab}}, \mathbb{C}) \longrightarrow \cdots \quad (21)$$

Tensoring this chain complex with the $\mathbb{C}[\pi_{\text{ab}}]$ -module \mathbb{C}_ρ , we obtain a chain complex of \mathbb{C} -vector spaces,

$$\cdots \longrightarrow C_{i+1}(X, \mathbb{C}_\rho) \xrightarrow{\partial_{i+1}^{\text{ab}}(\rho)} C_i(X, \mathbb{C}_\rho) \xrightarrow{\partial_i^{\text{ab}}(\rho)} C_{i-1}(X, \mathbb{C}_\rho) \longrightarrow \cdots, \quad (22)$$

where the evaluation of ∂_i^{ab} at ρ is obtained by applying the ring morphism $\mathbb{C}[\pi] \rightarrow \mathbb{C}$, $g \mapsto \rho(g)$ to each entry. Taking homology in degree i of this chain complex, we obtain the twisted homology groups $H_i(X, \mathbb{C}_\rho)$, whose jumps in dimension the variety $\mathcal{V}_i(X)$ keeps track of.

In a similar fashion, we may define the cohomology jump loci $\mathcal{V}^i(X)$ by the condition $H^i(X, \mathbb{C}_\rho) \neq 0$. Note that $H^i(X, \mathbb{C}_\rho) \cong H_i(X, \mathbb{C}_{\rho^{-1}})$; thus, the inversion automorphism $\rho \mapsto \rho^{-1}$ of the character group of X identifies $\mathcal{V}^i(X)$ with $\mathcal{V}_i(X)$, for each $i \geq 0$.

4.2 Some properties of the characteristic varieties

The sets $\mathcal{V}_i(X)$ are algebraic subsets of the character group $\text{Char}(X)$. Clearly, $1 \in \mathcal{V}_i(X)$ if and only if the i -th Betti number $b_i(X)$ is non-zero. In degree 0, we have $\mathcal{V}_0(X) = \{1\}$. In degree 1, the variety $\mathcal{V}_1(X)$ depends only on the fundamental group $\pi = \pi_1(X, x_0)$ —in fact, only on its maximal metabelian quotient, π/π'' —so we shall sometimes denote it as $\mathcal{V}_1(\pi)$.

The characteristic varieties are homotopy-type invariants of our space. More precisely, if $f: X \rightarrow Y$ is a homotopy equivalence, then the induced morphism between character group, $f^*: \text{Char}(Y) \rightarrow \text{Char}(X)$, restrict to an isomorphism $f^*: \mathcal{V}_i(Y) \xrightarrow{\cong} \mathcal{V}_i(X)$, for all $i \geq 0$; see [66] for more details.

If $p: Y \rightarrow X$ is a finite, regular cover, then the induced morphism between character groups, $p^*: \text{Char}(X) \hookrightarrow \text{Char}(Y)$, maps each characteristic variety $\mathcal{V}_i(X)$ into $\mathcal{V}_i(Y)$; see [21, 67] for details.

As noted in [56], the characteristic varieties behave well under finite direct products. More precisely, let X_1 and X_2 be two connected, finite-type CW-complexes. Identifying the character group of the product $X = X_1 \times X_2$ with $\text{Char}(X_1) \times \text{Char}(X_2)$, we have

$$\mathcal{V}_i(X_1 \times X_2) = \bigcup_{p+q=i} \mathcal{V}_p(X_1) \times \mathcal{V}_q(X_2). \quad (23)$$

The proof of this formula is straightforward: For each character $\rho = (\rho_1, \rho_2) \in \text{Char}(X)$, the chain complex $C_*(X, \mathbb{C}_\rho)$ decomposes as the tensor product of the chain complexes $C_*(X_1, \mathbb{C}_{\rho_1})$ and $C_*(X_2, \mathbb{C}_{\rho_2})$. Taking homology, we see that $H_i(X, \mathbb{C}_\rho) = \bigoplus_{p+q=i} H_p(X_1, \mathbb{C}_{\rho_1}) \otimes_{\mathbb{C}} H_q(X_2, \mathbb{C}_{\rho_2})$, and the claim follows.

4.3 Alexander varieties

An alternative approach, going back to the definition of the Alexander polynomials of knots and links [1], uses the homology modules of the universal abelian cover of our space X . As before, let $\pi = \pi_1(X)$, and let

$$H_i(X^{\text{ab}}, \mathbb{C}) = H_i(X, \mathbb{C}[\pi_{\text{ab}}]) \quad (24)$$

be the homology groups of the chain complex (21). These *Alexander invariants* are in a natural way modules over the group ring $\mathbb{C}[\pi_{\text{ab}}]$. Identifying this commutative, Noetherian ring with the coordinate ring of the character group of π , we let

$$\tilde{\mathcal{V}}_i(X) = \text{supp}(H_i(X^{\text{ab}}, \mathbb{C})) \quad (25)$$

be the subvariety of $\text{Char}(X)$ defined by the annihilator ideal of the respective homology module. One may also consider the cohomology modules $H^i(X^{\text{ab}}, \mathbb{C})$ and their support varieties, $\tilde{\mathcal{V}}^i(X)$; we will not pursue this approach here, but refer instead to [39] for details.

As shown in [56, 57], the characteristic varieties and their homology support loci counterparts are related in the following way:

$$\bigcup_{i \leq q} \mathcal{V}_i(X) = \bigcup_{i \leq q} \tilde{\mathcal{V}}_i(X). \quad (26)$$

Of special interest is the first characteristic variety, $\mathcal{V}_1(\pi) = \mathcal{V}_1(X)$. Suppose π admits a finite presentation, say, $F/R = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, and let $\phi: F \twoheadrightarrow \pi$ be the presenting homomorphism. Let $\partial_j r_i \in \mathbb{Z}[F]$ be the Fox derivatives of the relators, and let $\partial_2^{\text{ab}} = (\partial_j r_i)^{\text{ab} \circ \phi}$ be the corresponding Alexander matrix, with entries in $\mathbb{Z}[\pi_{\text{ab}}]$. It follows from (26) that $\mathcal{V}_1(\pi)$ coincides (at least away from 1) with the zero locus of the ideal $E_1(\pi)$ of codimension 1 minors of ∂_2^{ab} , a result due to E. Hironaka [36].

As shown in [47], the first resonance variety $\mathcal{R}_1(\pi)$ admits a similar description, at least when π is a commutator-relators group, i.e., when all the relators r_i belong to the commutator subgroup $[F, F]$. In this case, $\mathcal{R}_1(\pi)$ is the zero locus of the codimension 1 minors of the ‘linearized’ Alexander matrix, $(\partial_2^{\text{ab}})^{\text{lin}}$, which is the m by n matrix over the polynomial ring $S = \mathbb{Z}[y_1, \dots, y_n]$ with ij -entries equal to $\sum_{k=1}^n \epsilon(\partial_k \partial_j r_i) y_k$, where $\epsilon: \mathbb{Z}[F] \rightarrow \mathbb{Z}$ is the augmentation map.

Remark 4.1. The characteristic varieties can be arbitrarily complicated. For instance, let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be an integral Laurent polynomial. Then, as shown in [69], there is a finitely presented group π with $\pi_{\text{ab}} = \mathbb{Z}^n$ and $\mathcal{V}_1(\pi) = V(f) \cup \{1\}$.

More generally, let Z be an algebraic subset of $(\mathbb{C}^*)^n$, defined over \mathbb{Z} , and let k be a positive integer. Then, as shown in [72], there is a finite, connected CW-complex X with $\text{Char}(X) = (\mathbb{C}^*)^n$ such that $\mathcal{V}_i(X) = \{1\}$ for $i < q$ and $\mathcal{V}_q(X) = Z \cup \{1\}$. \diamond

5 The tangent cone theorem

We are now ready to state a key result in the theory of cohomology jump loci: given a space X , and an algebraic model A with good finiteness properties, the

characteristic varieties $\mathcal{V}^i(X) = \mathcal{V}_i(X)$ may be identified around the identity with the resonance varieties $\mathcal{R}^i(A)$. The resulting Tangent Cone theorem imposes strong restrictions on the nature of the resonance varieties $\mathcal{R}^i(X)$ of a formal space X .

5.1 Two types of tangent cones

We start by reviewing two constructions which yield approximations to a subvariety W of a complex algebraic torus $(\mathbb{C}^*)^n$. The first one is the classical tangent cone, a variant of the construction described in §3.3, while the second one is the exponential tangent cone, a construction first introduced in [23] and further studied in [66, 69].

Let I be an ideal in the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ such that $W = V(I)$. Picking a finite generating set for I , and multiplying these generators with suitable monomials if necessary, we see that W may also be defined by the ideal $I \cap R$ in the polynomial ring $R = \mathbb{C}[t_1, \dots, t_n]$. Let J be the ideal in the polynomial ring $S = \mathbb{C}[z_1, \dots, z_n]$, generated by the polynomials $g(z_1, \dots, z_n) = f(z_1 + 1, \dots, z_n + 1)$, for all $f \in I \cap R$.

The *tangent cone* of W at 1 is the algebraic subset $\text{TC}_1(W) \subset \mathbb{C}^n$ defined by the ideal $\text{in}(J) \subset S$ generated by the initial forms of all non-zero elements from J . As before, the set $\text{TC}_1(W)$ is a homogeneous subvariety of \mathbb{C}^n , which depends only on the analytic germ of W at the identity. In particular, $\text{TC}_1(W) \neq \emptyset$ if and only if $1 \in W$. Moreover, TC_1 commutes with finite unions.

On the other hand, the *exponential tangent cone* to W at the origin is the set

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \text{ for all } \lambda \in \mathbb{C}\}. \quad (27)$$

As shown in [23, 66], this set is a finite union of rationally defined linear subspaces of the affine space \mathbb{C}^n . An alternative interpretation of this construction is given in [69, §6.3].

It is readily seen that τ_1 commutes with finite unions and arbitrary intersections. Clearly, the exponential tangent cone of W only depends on the analytic germ of W at the identity $1 \in (\mathbb{C}^*)^n$. In particular, $\tau_1(W) \neq \emptyset$ if and only if $1 \in W$.

Example 5.1. Suppose W is an algebraic subtorus of $(\mathbb{C}^*)^n$. Then $\tau_1(W)$ equals $\text{TC}_1(W)$, and both coincide with $T_1(W)$, the tangent space at the identity to the Lie group W . \diamond

More generally, there is always an inclusion between the two types of tangent cones associated to an algebraic subset $W \subset (\mathbb{C}^*)^n$, namely,

$$\tau_1(W) \subseteq \text{TC}_1(W). \quad (28)$$

But, as we shall see in several examples spread through this paper, this inclusion is far from being an equality for an arbitrary W . For instance, the tangent cone $\mathrm{TC}_1(W)$ may be a non-linear, irreducible subvariety of \mathbb{C}^n , or $\mathrm{TC}_1(W)$ may be a linear space containing the exponential tangent cone $\tau_1(W)$ as a union of proper linear subspaces.

5.2 Germs of jump loci

As before, let X be a connected, finite-type CW-complex. Recall that, for each $i \geq 0$, we have a characteristic variety $\mathcal{V}^i(X)$ inside the abelian, complex algebraic group $\mathrm{Char}(X)$. Furthermore, the identity component of this algebraic group, $\mathrm{Char}(X)^0$, is isomorphic to $(\mathbb{C}^*)^n$, where $n = b_1(X)$.

Now suppose we have a finite-type CDGA model (A, d) for our space X . Then, for each $i \geq 0$, we have a resonance variety $\mathcal{R}^i(A)$ inside the affine space $H^1(A) = H^1(X, \mathbb{C})$. Furthermore, this affine space may be identified with \mathbb{C}^n , the tangent space at 1 to $(\mathbb{C}^*)^n$.

The next result, due to Dimca and Papadima [22], relates the two types of cohomology jump loci around the origins of the respective ambient spaces.

Theorem 5.2 ([22]). *Suppose Sullivan's model $A_{\mathrm{PL}}(X)$ is q -equivalent to a finite-type CDGA (A, d) . Then, for all $i \leq q$, the germ at 1 of $\mathcal{V}^i(X)$ is isomorphic to the germ at 0 of $\mathcal{R}^i(A)$.*

It is important to note that all the above isomorphisms are induced by an analytic isomorphism $\mathrm{Char}(X)_{(1)}^0 \cong H^1(A)_{(0)}$, the inverse of which is obtained by suitably restricting the exponential map $\exp: \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$.

Theorem 5.2 shows that, at least around the origin, the resonance varieties of a finite-type CDGA model for X depend only on the characteristic varieties of X , and thus, only on the homotopy type of X . This observation leads to the following corollary.

Corollary 5.3. *Let X be finite-type CW-complex. Suppose (A, d) and (A', d') are two finite-type CDGAs, both q -equivalent to the Sullivan model $A_{\mathrm{PL}}(X)$, for some $q \geq 1$. There is then an isomorphism $H^1(A) \cong H^1(A')$ restricting to isomorphisms $\mathcal{R}^i(A)_{(0)} \cong \mathcal{R}^i(A')_{(0)}$, for all $i \leq q$.*

A particular case of Theorem 5.2 is worth singling out.

Corollary 5.4. *If X is a q -formal space, then, for all $i \leq q$, the germ at 1 of $\mathcal{V}^i(X)$ is isomorphic to the germ at 0 of $\mathcal{R}^i(X)$.*

A precursor to this corollary can be found in the pioneering work of Green and Lazarsfeld [34, 35] on the cohomology jump loci of compact Kähler manifolds. The case when $q = 1$ was first established in [23, Theorem A]. For further developments in this direction, we refer to [10, 45].

5.3 Tangent cones and jump loci

Returning now to the general situation, consider an arbitrary connected, finite-type CW-complex X . We then have the following relationship (due to Libgober) between the characteristic and resonance varieties of such a space.

Theorem 5.5 ([41]). *For all $i \geq 0$,*

$$\mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X). \quad (29)$$

Putting together these inclusions with those from (28), we obtain the following immediate corollary.

Corollary 5.6. *For all $i \geq 0$,*

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X). \quad (30)$$

In particular, if $\mathcal{R}^i(X) = \{0\}$, then $\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \{0\}$.

In general, though, each of the inclusions from (30), or both inclusions can be strict, as examples to follow will show.

We now turn to spaces which admit finite-type algebraic models, and to the relations that hold between cohomology jump loci in this framework.

Theorem 5.7. *Let X be a connected, finite-type CW-complex, and suppose the Sullivan model $A_{\mathrm{PL}}(X)$ is q -equivalent to a finite-type CDGA A . Then, for all $i \leq q$,*

1. $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathrm{TC}_0(\mathcal{R}^i(A))$.
2. *If, moreover, A has positive weights, and the q -equivalence between $A_{\mathrm{PL}}(X)$ and A preserves \mathbb{Q} -structures, then $\mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A)$.*

Proof. The first assertion follows at once from Theorem 5.2. The second assertion follows from the first one, when coupled with Theorem 3.3. \square

The following examples show that the positive-weights assumption in Theorem 5.7, part 2 is really necessary. That is, we cannot always replace $\mathrm{TC}_0(\mathcal{R}^i(A))$ with $\mathcal{R}^i(A)$ in part 1.

Example 5.8. Let $X = S^1$. We can take as a finite-dimensional model for the circle the CDGA (A, d) from Example 2.9, with $A = \bigwedge(a, b)$ and $da = 0$, $db = b \cdot a$. Since $\mathcal{V}^1(S^1) = \{1\}$, the resonance variety $\mathcal{R}^1(A) = \{0, 1\}$ properly contains $\mathrm{TC}_1(\mathcal{V}^1(S^1)) = \{0\}$. Of course, we can also take as a model for S^1 its cohomology algebra, $A' = \bigwedge(a)$, endowed with the zero differential, in which case the conclusion of part 2 is satisfied. \diamond

Example 5.9. Let Γ be a discrete, co-compact subgroup of a simply-connected, solvable, real Lie group G , and let $M = G/\Gamma$ be the corresponding solvmanifold. As shown in [37, 53], all the characteristic varieties of M are

finite subsets of $\text{Char}(M)$. Moreover, as shown by Papadima and Păunescu in [53], if (A, d) is any finite-dimensional model for M (such as the one constructed by Kasuya [37]), then all the resonance varieties $\mathcal{R}^i(A)$ contain 0 as an isolated point; in particular, $\text{TC}_1(\mathcal{V}^i(M)) = \text{TC}_0(\mathcal{R}^i(A)) = \{0\}$.

Now suppose G is completely solvable, and (A, d) is the classical Hattori model for the solvmanifold $M = G/\Gamma$. Work of Millionschikov [49], as reprised in [53], shows that $\mathcal{R}^i(A)$ is also a finite set. Furthermore, there are examples of solvmanifolds of this type where $\mathcal{R}^i(A)$ is different from $\{0\}$. \diamond

5.4 The influence of formality

The main connection between the formality property of a space and its cohomology jump loci is provided by the following theorem. (Again, the case $q = 1$ was established in [23], and the general case in [22].)

Theorem 5.10 ([23, 22]). *If X is a q -formal space, the following “tangent cone formula” holds, for all $i \leq q$,*

$$\pi_1(\mathcal{V}^i(X)) = \text{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X). \quad (31)$$

As an application of this theorem, we have the following characterization of the irreducible components of the cohomology jump loci in the formal setting.

Corollary 5.11. *If X is a q -formal space, then, for all $i \leq q$,*

1. *All irreducible components of the resonance variety $\mathcal{R}^i(X)$ are rationally defined subspaces of $H^1(X, \mathbb{C})$.*
2. *All irreducible components of the characteristic variety $\mathcal{V}^i(X)$ which pass through the origin are algebraic subtori of $\text{Char}(X)^0$, of the form $\exp(L)$, where L runs through the linear subspaces comprising $\mathcal{R}^i(X)$.*

Even when the space X is formal, the characteristic varieties $\mathcal{V}^i(X)$ may have irreducible components which do not pass through the identity of $\text{Char}(X)^0$, and thus are not detected by the resonance varieties $\mathcal{R}^i(X)$.

Example 5.12. Let K be a non-trivial knot in the 3-sphere, with complement $X = S^3 \setminus K$. Then $H^*(X, \mathbb{Z}) \cong H^*(S^1, \mathbb{Z})$; therefore, X is formal and $\mathcal{R}^1(X) = \{0\}$. The characteristic variety $\mathcal{V}^1(X) \subset \mathbb{C}^*$ consists of 1, together with all the roots of the Alexander polynomial, $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. Thus, if $\Delta_K \neq 1$, then $\mathcal{V}^1(X)$ has components which do not contain 1. \diamond

Example 5.13. Let X be the 2-complex obtained by gluing a Möbius band to a 2-torus along a meridian circle. Then X has the same rational cohomology ring as the 2-torus; thus, X is formal and $\mathcal{R}^1(X) = \{0\}$. On the other hand, $\pi_1(X) = \langle x_1, x_2 \mid x_1 x_2^2 = x_2^2 x_1 \rangle$; hence, the variety $\mathcal{V}^1(X) \subset (\mathbb{C}^*)^2$ consists of the identity together with the translated subtorus $t_1 t_2^{-1} = -1$. \diamond

5.5 Formality tests

The next several examples illustrate the various ways in which the inclusions from Corollary 5.6 may fail to hold as equalities, thereby showing how Theorem 5.10 and Corollary 5.11 can be used to detect non-formality. More examples will be given in Sections 6–8.

Example 5.14. Let X be the presentation 2-complex for the group $\pi = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] \rangle$. In this case, $\mathcal{V}^1(X) = \{t_1 = 1\}$, and so $\tau_1(\mathcal{V}^1(X)) = \text{TC}_1(\mathcal{V}^1(X)) = \{x_1 = 0\}$. On the other hand, $\mathcal{R}^1(X) = \mathbb{C}^2$, and so X is not 1-formal. \diamond

The next example (adapted from [23]) shows how the rationality property from Corollary 5.11 can be used as a formality test.

Example 5.15. Let X be the presentation 2-complex for the group π with generators x_1, \dots, x_4 and relators $r_1 = [x_1, x_2]$, $r_2 = [x_1, x_4][x_2^{-2}, x_3]$, and $r_3 = [x_1^{-1}, x_3][x_2, x_4]$. Computing the linearized Alexander matrix of this presentation by the method described in §4.3, we see that $\mathcal{R}^1(X)$ is the quadric hypersurface in \mathbb{C}^4 given by the equation $z_1^2 - 2z_2^2 = 0$. This quadric splits into two linear subspaces defined over \mathbb{R} , but it is irreducible over \mathbb{Q} . Thus, X is not 1-formal. \diamond

Example 5.16. In view of Remark 4.1, there is a finitely presented group π with abelianization \mathbb{Z}^3 and characteristic variety

$$\mathcal{V}^1(\pi) = \{(t_1, t_2, t_3) \in (\mathbb{C}^*)^3 \mid (t_2 - 1) = (t_1 + 1)(t_3 - 1)\}.$$

As noted in [69], this variety is irreducible, and its exponential tangent cone at the origin splits as a union of two (rationally defined) lines in \mathbb{C}^3 ,

$$\tau_1(\mathcal{V}^1(\pi)) = \{x_2 = x_3 = 0\} \cup \{x_1 - x_3 = x_2 - 2x_3 = 0\}.$$

The variety $\mathcal{V}^1(\pi)$ is a complex, 2-dimensional torus passing through the origin. Nevertheless, this torus does not embed as an algebraic subgroup in $(\mathbb{C}^*)^3$; indeed, if it did, $\tau_1(\mathcal{V}^1(\pi))$ would be a single plane. Consequently, the group π is not 1-formal. \diamond

6 Smooth quasi-projective varieties

We now switch our focus, from the general theory of cohomology jump loci to some of the applications of this theory within the class of smooth, complex quasi-projective varieties. For such spaces, the cohomology jump loci are severely restricted by the extra structure imposed on their CDGA models and cohomology rings by the underlying algebraic geometry.

6.1 Compactifications and formality

A complex projective variety is a subset of a complex projective space \mathbb{CP}^n , defined as the zero-locus of a homogeneous prime ideal in $\mathbb{C}[z_0, \dots, z_n]$. A Zariski open subvariety of a projective variety is called a quasi-projective variety. We will only consider here projective and quasi-projective varieties which are connected and smooth.

If M is a (compact, smooth) projective variety, then the Hodge decomposition on $H^*(M, \mathbb{C})$ puts strong constraints on the topological properties of M . For instance, as shown in [15], every such a manifold is formal.

Each smooth, quasi-projective variety X admits a good compactification. That is to say, there is a smooth, complex projective variety \overline{X} and a normal-crossings divisor D such that $X = \overline{X} \setminus D$. By a well-known theorem of Deligne, each cohomology group of X admits a mixed Hodge structure. This additional structure puts definite constraints on the algebraic topology of such manifolds.

For instance, if X admits a smooth compactification \overline{X} with $b_1(\overline{X}) = 0$, the weight 1 filtration on $H^1(X, \mathbb{C})$ vanishes; in turn, by work of Morgan [51], this implies the 1-formality of X . Thus, as noted by Kohno in [38], if X is the complement of a hypersurface in \mathbb{CP}^n , then $\pi_1(X)$ is 1-formal. Moreover, if $n = 2$, then X itself is formal, see [11, 44].

In general, though, smooth quasi-projective varieties need not be 1-formal. Furthermore, even when they are 1-formal, they still can be non-formal.

Example 6.1. Let $E^{\times n}$ be the n -fold product of an elliptic curve. The closed form $\frac{1}{2}\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ defines an integral cohomology class $\omega \in H^{1,1}(E^{\times n})$. By the Lefschetz theorem on $(1, 1)$ -classes, ω can be realized as the first Chern class of an algebraic line bundle over $E^{\times n}$. Let X_n be the complement of the zero-section of this bundle. Then X_n is a smooth, quasi-projective variety which is not formal. In fact, as we shall see in Example 6.7, X_1 is not 1-formal. On the other hand, X_n is 1-formal, for all $n > 1$. \diamond

6.2 Algebraic models

As before, let X be a connected, smooth quasi-projective variety, and choose a smooth compactification \overline{X} such that the complement is a finite union, $D = \bigcup_{j \in J} D_j$, of smooth divisors with normal crossings. There is then a rationally defined CDGA, $A = A(\overline{X}, D)$, called the *Gysin model* of the compactification, constructed as follows. As a \mathbb{C} -vector space, A^i is the direct sum of all subspaces

$$A^{p,q} = \bigoplus_{|S|=q} H^p\left(\bigcap_{k \in S} D_k, \mathbb{C}\right)(-q) \quad (32)$$

with $p + q = i$, where $(-q)$ denotes the Tate twist. Furthermore, the multiplication in A is induced by the cup-product in \overline{X} , and has the property that $A^{p,q} \cdot A^{p',q'} \subseteq A^{p+p',q+q'}$, while the differential, $d: A^{p,q} \rightarrow A^{p+2,q-1}$, is constructed from the Gysin maps arising from intersections of divisors. The CDGA just constructed depends on the compactification \overline{X} ; for simplicity, though, we will denote it by $A(X)$ when the compactification is understood.

An important particular case is when our variety X has dimension 1. That is to say, let Σ be a connected, possibly non-compact, smooth algebraic curve. Then Σ admits a canonical compactification, $\overline{\Sigma}$, and thus, a canonical Gysin model, $A(\Sigma)$. We illustrate the construction of this model in a simple situation, which we shall encounter again in Section 8.

Example 6.2. Let $\Sigma = E^*$ be a once-punctured elliptic curve. Then $\overline{\Sigma} = E$, and the Gysin model $A(\Sigma)$ is the algebra $A = \bigwedge(a, b, e)/(ae, be)$ on generators a, b in bidegree $(1, 0)$ and generator e in bidegree $(0, 1)$, with differential $d: A \rightarrow A$ given by $da = db = 0$ and $de = ab$. \diamond

The above construction is functorial, in the following sense: If $f: X \rightarrow Y$ is a morphism of quasi-projective manifolds which extends to a regular map $\bar{f}: \overline{X} \rightarrow \overline{Y}$ between the respective good compactifications, then there is an induced CDGA morphism $f^!: A(Y) \rightarrow A(X)$ which respects the bigradings.

Morgan showed in [51] that the Sullivan model $A_{PL}(X)$ is connected to the Gysin model $A(X)$ by a chain of quasi-isomorphisms preserving \mathbb{Q} -structures. Moreover, setting the weight of $A^{p,q}$ equal to $p + 2q$ defines a positive-weight decomposition on (A^*, d) .

In [25], Dupont constructs a Gysin-type model for certain types of quasi-projective varieties, where the normal-crossing divisors assumption on the compactification can be relaxed. More precisely, let \mathcal{A} be an arrangement of smooth hypersurfaces in a smooth, n -dimensional complex projective variety \overline{X} , and suppose \mathcal{A} locally looks like an arrangement of hyperplanes in \mathbb{C}^n . There is then a CDGA model for the complement, $X = \overline{X} \setminus \bigcup_{L \in \mathcal{A}} L$, which builds on the combinatorial definition of the Orlik–Solomon algebra of a hyperplane arrangement (an algebra we will return to in §7.1).

6.3 Configuration spaces

In a special situation, an alternate model is available. A construction due to Fadell and Neuwirth associates to a space X and a positive integer n the space of ordered configurations of n points in X ,

$$\text{Conf}(X, n) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}. \quad (33)$$

The E_2 -term of the Leray spectral sequence for the inclusion $\text{Conf}(X, n) \hookrightarrow X^n$ was described concretely by Cohen and Taylor in the late 1970s. If X is

a smooth, complex projective variety of dimension m , then $\text{Conf}(X, n)$ is a smooth, quasi-projective variety. Moreover, as shown by Totaro in [71], the Cohen–Taylor spectral sequence collapses at the E_{m+1} -term, and the E_m -term is a CDGA model for the configuration space $\text{Conf}(X, n)$.

The most basic example is the configuration space of n ordered points in \mathbb{C} , which is a classifying space for P_n , the pure braid group on n strings, whose cohomology ring was computed by Arnol'd in the late 1960s. We shall come back to this example in §7 in the setting of arrangements of hyperplanes, and we shall look at configuration space of n points on an elliptic curve E in §8, in the setting of elliptic arrangements.

More generally, following Eastwood and Huggett [26], one may consider the “graphic configuration spaces”

$$\text{Conf}(X, \Gamma) = \{(x_1, \dots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ for } \{i, j\} \in E(\Gamma)\} \quad (34)$$

associated to a space X and a simple graph Γ with vertex set $[n]$ and edge set $E(\Gamma)$. Especially interesting is the case when X is a Riemann surface Σ_g . For such a space, the naive compactification, $\overline{\text{Conf}}(X, n) = X^{\times n}$, satisfies the hypothesis which permit the construction of the Dupont model, [25]. For recent work exploiting this model, we refer to [5].

6.4 Characteristic varieties

The structure of the jump loci for cohomology in rank 1 local systems on smooth, complex projective and quasi-projective varieties (and, more generally, on Kähler and quasi-Kähler manifolds) was determined through the work of Beauville [4], Green and Lazarsfeld [34, 35], Simpson [61], and Arapura [2].

In the quasi-projective setting, further improvements and refinements were given in [40, 3, 66]. The definitive structural result was obtained by Budur and Wang in [9], building on work of Dimca and Papadima [22].

Theorem 6.3 ([9]). *Let X be a smooth quasi-projective variety. Then each characteristic variety $\mathcal{V}^i(X)$ is a finite union of torsion-translated subtori of $\text{Char}(X)$.*

Work of Arapura [2] explains how the non-translated subtori occurring in the above decomposition of $\mathcal{V}^1(X)$ arise. Let us say that a holomorphic map $f: X \rightarrow \Sigma$ is *admissible* if f is surjective, has connected generic fiber, and the target Σ is a connected, smooth complex curve with negative Euler characteristic. Up to reparametrization at the target, the variety X admits only finitely many admissible maps; let \mathcal{E}_X be the set of equivalence classes of such maps.

If $f: X \rightarrow \Sigma$ is an admissible map, it is readily verified that $\mathcal{V}^1(\Sigma) = \text{Char}(\Sigma)$. Thus, the image of the induced morphism between character groups, $f^*: \text{Char}(\Sigma) \rightarrow \text{Char}(X)$, is an algebraic subtorus of $\text{Char}(X)$.

Theorem 6.4 ([2]). *The correspondence $f \mapsto f^*(\text{Char}(\Sigma))$ establishes a bijection between the set \mathcal{E}_X of equivalence classes of admissible maps from X to curves and the set of positive-dimensional, irreducible components of $\mathcal{V}^1(X)$ containing 1.*

The positive-dimensional, irreducible components of $\mathcal{V}^1(X)$ which do not pass through 1 can be similarly described, by replacing the admissible maps with certain “orbifold fibrations,” whereby multiple fibers are allowed. For more details and further explanations, we refer to [3, 66].

6.5 Resonance varieties

We now turn to the resonance varieties associated to a quasi-projective manifold, and how they relate to the characteristic varieties. The Tangent Cone theorem takes a very special form in this algebro-geometric setting.

Theorem 6.5. *Let X be a smooth, quasi-projective variety, and let $A(X)$ be a Gysin model for X . Then, for each $i \geq 0$,*

$$\tau_1(\mathcal{V}^i(X)) = \text{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(A(X)) \subseteq \mathcal{R}^i(X). \quad (35)$$

Moreover, if X is q -formal, the last inclusion is an equality, for all $i \leq q$.

Proof. By Theorem 6.3, each irreducible component of $\mathcal{V}^i(X)$ passing through 1 is a complex algebraic subtorus $W \subset \text{Char}(X)$. As noted in Example 5.1, $\tau_1(W) = \text{TC}_1(W)$. Since both τ_1 and TC_1 commute with finite unions, the first equality in (35) follows.

Next, recall that $A(X)$ is a finite-type, rationally defined CDGA which admits positive weights. Moreover, there is a weak equivalence between $A(X)$ and $A_{\text{PL}}(X)$ preserving the respective \mathbb{Q} -structures. The second equality now follows from Theorem 5.7, part 2, while the last inclusion follows from Theorem 3.3. \square

In particular, the resonance varieties $\mathcal{R}^i(A(X))$ are finite unions of rationally defined linear subspaces of $H^1(X, \mathbb{C})$. On the other hand, the varieties $\mathcal{R}^i(X)$ can be much more complicated: for instance, they may have non-linear irreducible components. If X is q -formal, though, Theorem 35 guarantees this cannot happen, as long as $i \leq q$.

6.6 Resonance in degree 1

Once again, let X be a smooth, quasi-projective variety, and let $A(X)$ be the Gysin model associated to a good compactification \overline{X} . The degree 1 resonance

varieties $\mathcal{R}^1(A(X))$, and, to some extent, $\mathcal{R}^1(X)$, admit a much more precise description than those in higher degrees.

As in the setup from Theorem 6.4, let \mathcal{E}_X be the set of equivalence classes of admissible maps from X to curves, and let $f: X \rightarrow \Sigma$ be such map. Recall from §6.2 that the curve Σ admits a canonical Gysin model, $A(\Sigma)$. As noted in [22], the induced CDGA morphism, $f^!: A(\Sigma) \rightarrow A(X)$, is injective. Let $f^*: H^1(A(\Sigma)) \rightarrow H^1(A(X))$ be the induced homomorphism in cohomology.

Theorem 6.6 ([22, 45]). *For a smooth, quasi-projective variety X , the decomposition of $\mathcal{R}^1(A(X))$ into (linear) irreducible components is given by*

$$\mathcal{R}^1(A(X)) = \bigcup_{f \in \mathcal{E}_X} f^*(H^1(A(\Sigma))). \quad (36)$$

If X admits no admissible maps, i.e., $\mathcal{E}_X = \emptyset$, formula (36) should be understood to mean $\mathcal{R}^1(A(X)) = \{0\}$ if $b_1(X) > 0$ and $\mathcal{R}^1(A(X)) = \emptyset$ if $b_1(X) = 0$.

Example 6.7. Let $X = X_1$ be the complex, smooth quasi-projective surface constructed in Example 6.1. Clearly, this manifold is a \mathbb{C}^* -bundle over $E = S^1 \times S^1$ which deform-retracts onto the Heisenberg manifold from Example 3.5. Hence, $\mathcal{V}^1(X) = \{1\}$, and so $\tau_1(\mathcal{V}^1(X)) = \mathrm{TC}_1(\mathcal{V}^1(X)) = \{0\}$. On the other hand, $\mathcal{R}^1(X) = \mathbb{C}^2$, and so X is not 1-formal. \diamond

Under a 1-formality assumption, the usual resonance varieties $\mathcal{R}^1(X)$ admit a similar description.

Theorem 6.8 ([23]). *Let X be a smooth, quasi-projective variety, and suppose X is 1-formal. The decomposition into irreducible components of the first resonance variety is then given by*

$$\mathcal{R}^1(X) = \bigcup_{f \in \mathcal{E}_X} f^*(H^1(\Sigma, \mathbb{C})), \quad (37)$$

with the same convention as before when $\mathcal{E}_X = \emptyset$. Moreover, all the (rationally defined) linear subspaces in this decomposition have dimension at least 2, and any two distinct ones intersect only at 0.

If X is compact, then the formality assumption in the above theorem is automatically satisfied, due to [15]. Furthermore, the conclusion of the theorem can also be sharpened in this case: each (non-trivial) irreducible component of $\mathcal{R}^1(X)$ is even-dimensional, of dimension at least 4.

In general, though, the resonance varieties $\mathcal{R}^1(X)$ can have non-linear components. For instance, if $X = \mathrm{Conf}(E, n)$ is the configuration space of $n \geq 3$ points on an elliptic curve E , then $\mathcal{R}^1(X)$ is an irreducible, non-linear variety (in fact, a rational normal scroll), see [23] and also Example 8.4 below.

7 Hyperplane arrangements and the Milnor fibration

Next, we turn our focus to a class of quasi-projective varieties which are obtained by deleting finitely many hyperplanes from a complex affine space. These hyperplane arrangement complements are formal spaces, yet the associated Milnor fibers may fail the Tangent Cone test for formality.

7.1 Complement and intersection lattice

A hyperplane arrangement \mathcal{A} is a finite collection of codimension 1 linear subspaces in a complex affine space \mathbb{C}^n . Its complement, $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$, is a connected, smooth, quasi-projective variety. This manifold is a Stein domain, and thus has the homotopy-type of a finite CW-complex of dimension n . Moreover, $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$, where $U(\mathcal{A})$ is the complement in \mathbb{CP}^{n-1} of the projectivized arrangement.

The topological invariants of the complement are intimately tied to the combinatorics of the arrangement. The latter is encoded in the *intersection lattice*, $L(\mathcal{A})$, which is the poset of all intersections of \mathcal{A} , ordered by reverse inclusion. The rank of the arrangement, denoted $\text{rk}(\mathcal{A})$, is the codimension of the intersection $\Sigma(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} H$.

Example 7.1. The braid arrangement of rank $n - 1$ consists of the diagonal hyperplanes $H_{ij} = \{z_i - z_j = 0\}$ in \mathbb{C}^n . The complement of this arrangement is the configuration space $\text{Conf}(\mathbb{C}, n)$, while the intersection lattice is the lattice of partitions of $[n] = \{1, \dots, n\}$, ordered by refinement. \diamond

For each hyperplane $H \in \mathcal{A}$, pick a linear form $f_H \in \mathbb{C}[z_0, \dots, z_n]$ such that $\ker(f_H) = H$. The homogeneous polynomial $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$, then, is a defining polynomial for the arrangement.

Building on work of Arnol'd on the cohomology ring of $\text{Conf}(\mathbb{C}, n)$, Brieskorn showed in [8] that the closed 1-forms df_H/f_H generate the de Rham cohomology of $M(\mathcal{A})$. Moreover, the inclusion of the subalgebra generated by those forms into the de Rham algebra $\Omega_{\text{dR}}^*(M(\mathcal{A}))$ induces an isomorphism in cohomology; consequently, the complement $M(\mathcal{A})$ is a formal space.

In [52], Orlik and Solomon gave a simple combinatorial description of the cohomology ring of the complement. Let $E = \bigwedge(\mathcal{A})$ be the exterior algebra (over \mathbb{Z}) on degree-one classes e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$, and set $e_{\mathcal{B}} = \prod_{H \in \mathcal{B}} e_H$ for each sub-arrangement $\mathcal{B} \subset \mathcal{A}$. Next, define a differential $\partial: E \rightarrow E$ of degree -1 , starting from $\partial(e_H) = 1$, and extending it to a linear map on E , using the graded Leibniz rule. Then

$$H^*(M(\mathcal{A}), \mathbb{Z}) = \bigwedge(\mathcal{A})/I(\mathcal{A}), \quad (38)$$

where $I(\mathcal{A})$ is the (homogeneous) ideal generated by all elements of the form

$$\begin{aligned} e_{\mathcal{B}}, & \quad \text{if } \Sigma(\mathcal{B}) = \emptyset, \\ \partial e_{\mathcal{B}}, & \quad \text{if } \operatorname{codim} \Sigma(\mathcal{B}) < |\mathcal{B}|. \end{aligned} \tag{39}$$

More generally, suppose \mathcal{A} is an arrangement of complex linear subspaces in \mathbb{C}^n . Using a blow-up construction, De Concini and Procesi gave in [14] a ‘wonderful’ CDGA model for the complement of such an arrangement. Based on a simplification of this model due to Yuzvinsky [73], Feichtner and Yuzvinsky showed in [29] the following: If $L(\mathcal{A})$ is a geometric lattice, then the complement of \mathcal{A} is a formal space. In general, however, the complement of a complex subspace arrangement need not be formal. This phenomenon is illustrated in [16], within the class of coordinate subspace arrangements, and in [48], within the class of diagonal subspace arrangements.

7.2 Cohomology jump loci of the complement

Once again, let \mathcal{A} be a complex hyperplane arrangement. The resonance varieties of the arrangement, $\mathcal{R}^i(\mathcal{A}) := \mathcal{R}^i(M(\mathcal{A}))$, live inside the affine space $H^1(M(\mathcal{A}), \mathbb{C}) = \mathbb{C}^{|\mathcal{A}|}$. These varieties depend only on the Orlik–Solomon algebra of \mathcal{A} , and thus, only on the intersection lattice $L(\mathcal{A})$.

In [27], Falk asked whether the resonance varieties $\mathcal{R}^i(\mathcal{A})$ are finite unions of linear subspaces. A special case of the Tangent Cone theorem, proved in [13] specifically for arrangement complements and in degree $i = 1$, led to a positive answer to this question, at least for $\mathcal{R}^1(\mathcal{A})$. With the technology provided by the general version of the Tangent Cone theorem, it is now easy to answer Falk’s question in full generality. Indeed, since the complement $M(\mathcal{A})$ is a formal space, Corollary 5.11 shows that $\mathcal{R}^i(\mathcal{A})$ is, in fact, a finite union of rationally defined linear subspaces, for each $i \geq 0$.

In degree $i = 1$, these linear spaces can be described much more precisely. Indeed, as shown by Falk and Yuzvinsky in [28] (see also [43, 46]), each component of $\mathcal{R}^1(\mathcal{A})$ corresponds to a multinet on a sub-arrangement of \mathcal{A} .

Briefly, a k -*multinet* on \mathcal{A} is a partition into $k \geq 3$ subsets \mathcal{A}_α , together with an assignment of multiplicities m_H to each $H \in \mathcal{A}$, and a choice of rank 2 flats, called the base locus. All these data must satisfy certain compatibility conditions. For instance, any two hyperplanes from different parts of the partition intersect in the base locus, while the sum of the multiplicities over each part is constant. Furthermore, if X is a flat in the base locus, then the sum $n_X = \sum_{H \in \mathcal{A}_\alpha \cap \mathcal{A}_X} m_H$ is independent of α . The multinet is *reduced* if all the m_H ’s are equal to 1. If, moreover, all the n_X ’s are equal to 1, the multinet is, in fact, a *net*, a classical notion from combinatorial geometry.

Every k -multinet on \mathcal{A} gives rise to an admissible map $M(\mathcal{A}) \rightarrow \Sigma$, where $\Sigma = \mathbb{CP}^1 \setminus \{k \text{ points}\}$, and the converse also holds. Moreover, the set $\mathcal{E}_{M(\mathcal{A})}$ of admissible maps (up to reparametrization at the target) from $M(\mathcal{A})$ to curves coincides with the set of multinets (up to the natural S_k -permutation

action on k -multinets) on subarrangements of \mathcal{A} , see [28, 60]. The *essential* components of $\mathcal{R}^1(\mathcal{A})$ are those corresponding to multinets fully supported on \mathcal{A} .

Example 7.2. Let \mathcal{A} be a generic 3-slice of the braid arrangement of rank 3, with defining polynomial $Q(\mathcal{A}) = z_0 z_1 z_2 (z_0 - z_1)(z_0 - z_2)(z_1 - z_2)$. Take a generic plane section, and label the corresponding lines as 1 to 6. Then, the variety $\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 ‘local’ components, corresponding to the triple points 124, 135, 236, 456, and one essential component, corresponding to the 3-net (16|25|34). \diamond

From Theorem 6.4, we know that the characteristic varieties $\mathcal{V}^i(\mathcal{A}) := \mathcal{V}^i(M(\mathcal{A}))$ consists of subtori in $(\mathbb{C}^*)^n$, possibly translated by roots of unity, together with a finite number of torsion points. By Theorem 5.10, we have that $\text{TC}_1(\mathcal{V}^i(\mathcal{A})) = \mathcal{R}^i(\mathcal{A})$. Thus, the components of $\mathcal{V}^i(\mathcal{A})$ passing through the origin are completely determined by $L(\mathcal{A})$.

As pointed out in [62], though, the characteristic variety $\mathcal{V}^1(\mathcal{A})$ may contain translated subtori, that is, components not passing through 1. It is still not known whether such components are combinatorially determined.

7.3 The Milnor fibration

Once again, let \mathcal{A} be a hyperplane arrangement in \mathbb{C}^n , with complement $M = M(\mathcal{A})$ and defining polynomial $Q = Q(\mathcal{A})$. As shown by Milnor [50] in a more general context, the restriction of the polynomial map $Q: \mathbb{C}^n \rightarrow \mathbb{C}$ to the complement is a smooth fibration, $Q: M \rightarrow \mathbb{C}^*$.

The typical fiber of this fibration, $Q^{-1}(1)$, is called the *Milnor fiber* of the arrangement, and is denoted by $F = F(\mathcal{A})$. The Milnor fiber is a Stein domain of complex dimension n , and thus has the homotopy type of a finite CW-complex of dimension n . Furthermore, the monodromy homeomorphism $h: F \rightarrow F$ is given by $h(z) = e^{2\pi i/m} z$, where $m = |\mathcal{A}|$, and thus has order m .

Example 7.3. The Boolean arrangement consists of the coordinate hyperplanes in \mathbb{C}^n ; its complement is the complex algebraic torus $(\mathbb{C}^*)^n$. The map $Q: (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$, $z \mapsto z_1 \cdots z_n$ is a morphism of algebraic groups. Hence, the Milnor fiber $F = \ker Q$ is an algebraic subgroup, isomorphic to $(\mathbb{C}^*)^{n-1}$. \diamond

Example 7.4. Consider a pencil of m lines in \mathbb{C}^2 , with defining polynomial $Q = z_1^m - z_2^m$ and complement $M = \mathbb{C}^* \times \mathbb{C} \setminus \{m \text{ points}\}$. The Milnor fiber, then, is a smooth complex curve of genus $\binom{m-1}{2}$ with m punctures. \diamond

In general, though, the polynomial map $Q: \mathbb{C}^n \rightarrow \mathbb{C}$ will have a non-isolated singularity at 0, and the topology of the Minor fiber $F = F(\mathcal{A})$ will be much more difficult to ascertain. In particular, it is a long-standing open problem to decide whether the first Betti number $b_1(F)$ is determined by the

intersection lattice of \mathcal{A} , and, if so, to find an explicit combinatorial formula for it, see for instance [12, 17, 20, 21, 42, 67]. Marked progress towards a positive solution to this problem was made recently in [60], using in an essential way the relationship between the varieties $\mathcal{V}^1(\mathcal{A})$ and $\mathcal{R}^1(\mathcal{A})$ provided by the Tangent Cone theorem, as well as the multinet interpretation of the components of $\mathcal{R}^1(\mathcal{A})$.

To make this machinery work, one starts by viewing the Minor fiber F as the regular, cyclic m -fold cover of the projectivized complement, $U = \mathbb{P}(M)$, defined by the homomorphism $\delta: \pi_1(U) \rightarrow \mathbb{Z}_m$ which takes each meridian generator to 1, see [12] and also [63, 67]. Embedding \mathbb{Z}_m into \mathbb{C}^* by sending $1 \mapsto e^{2\pi i/m}$, we may view δ as a character on $\pi_1(U)$. The relative position of this character with respect to the variety $\mathcal{V}^1(U) \cong \mathcal{V}^1(\mathcal{A})$ determines the first Betti number of F , as well as the characteristic polynomial of the algebraic monodromy, $h_*: H_1(F, \mathbb{C}) \rightarrow H_1(F, \mathbb{C})$.

7.4 Cohomology jump loci of the Milnor fiber

Very little is known about the homology with coefficients in rank 1 local systems of the Milnor fiber of an arrangement \mathcal{A} . Since $M(\mathcal{A})$ is a smooth, quasi-projective variety, Theorem 6.3 guarantees that the characteristic varieties $\mathcal{V}^i(F(\mathcal{A}))$ are finite unions of torsion-translated subtori.

Let $\pi: F(\mathcal{A}) \rightarrow U(\mathcal{A})$ be the restriction of the Hopf fibration to the Milnor fiber. Since π is a finite, regular cover, we have that $\pi^*(\mathcal{V}^i(U(\mathcal{A}))) \subseteq \mathcal{V}^i(F(\mathcal{A}))$. In general, though, this inclusion may well be strict. For instance, suppose \mathcal{A} admits a non-trivial, reduced multinet, and let T be the corresponding component of $\mathcal{V}^1(\mathcal{A})$. Then, as shown in [21], the variety $\mathcal{V}^1(F(\mathcal{A}))$ has an irreducible component passing through the identity and containing $\pi^*(T)$ as a proper subset.

Example 7.5. Let \mathcal{A} be the braid arrangement from Example 7.2. Recall that $\mathcal{V}^1(\mathcal{A})$ has four 2-dimensional components, T_1, \dots, T_4 , corresponding to the triple points, and also an essential, 2-dimensional component T , corresponding to a 3-net. The characteristic variety $\mathcal{V}^1(F(\mathcal{A})) \subset (\mathbb{C}^*)^7$ has four 2-dimensional components, $\pi^*(T_1), \dots, \pi^*(T_4)$, as well as 4-dimensional component W which properly includes the 2-torus $\pi^*(T)$. \diamond

Returning to the general situation, let again \mathcal{A} be a complex hyperplane arrangement, and let $F = F(\mathcal{A})$ be its Milnor fiber. By Theorem 6.5, $\text{TC}_1(\mathcal{V}^i(F)) = \mathcal{R}^i(A(F))$, where $A(F)$ is a Gysin model for F . Thus, to better understand the topology of the Milnor fiber, it would help a lot to address the following two problems.

Problem 7.6. Find a smooth compactification \overline{F} such that $\overline{F} \setminus F$ is a normal-crossings divisor. Does the monodromy $h: F \rightarrow F$ extend to a diffeomorphism $\bar{h}: \overline{F} \rightarrow \overline{F}$?

Problem 7.7. Given a compactification \overline{F} as above, write down an explicit presentation for the resulting Gysin model, $A(F)$. Furthermore, compute the resonance varieties $\mathcal{R}^i(A(F))$, and decide whether these varieties depend only on the intersection lattice $L(\mathcal{A})$.

7.5 Formality of the Milnor fiber

The following question was raised in [55], in a more general context: Is the Milnor fiber of a hyperplane arrangement \mathcal{A} always formal? Of course, if $\text{rank}(\mathcal{A}) = 2$, then $F(\mathcal{A})$ has the homotopy type of a wedge of circles, and so it is formal.

If $\text{rank}(\mathcal{A}) = 3$, formality and 1-formality are equivalent for the Milnor fiber, since in this case $F(\mathcal{A})$ has the homotopy type of a 2-complex. As noted by Dimca and Papadima [21], if the monodromy map acts as the identity on $H_1(F(\mathcal{A}), \mathbb{C})$, then $F(\mathcal{A})$ is formal. In general, though, the Milnor fiber of an arrangement is not formal, as the following example of Zuber [74] shows.

Example 7.8. Let \mathcal{A} be the arrangement associated to the complex reflection group $G(3, 3, 3)$, and defined by the polynomial $Q = (z_1^3 - z_2^3)(z_1^3 - z_3^3)(z_2^3 - z_3^3)$. The resonance variety $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^9$ has 12 local components, corresponding to the triple points, and 4 essential components corresponding to 3-nets.

Consider the 3-net whose associated rational map $\mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1$ is given by $(z_1, z_2, z_3) \mapsto (z_1^3 - z_2^3, z_2^3 - z_3^3)$. This map restricts to an admissible map $U(\mathcal{A}) \rightarrow \mathbb{CP}^1 \setminus \{(1, 0), (0, 1), (1, -1)\}$. Let T be the essential, 2-dimensional component of $\mathcal{V}_1(U(\mathcal{A}))$ obtained by pullback along this pencil. Further pulling back T via the covering projection $\pi: F(\mathcal{A}) \rightarrow U(\mathcal{A})$ produces a 4-dimensional subtorus inside $\text{Char}(F(\mathcal{A})) = (\mathbb{C}^*)^{12}$.

The subtorus $\pi^*(T)$ is of the form $\exp(L)$, for some linear subspace $L \subset H^1(F(\mathcal{A}), \mathbb{C})$. Using the mixed Hodge structure on the cohomology of the Milnor fiber, Zuber showed in [74] that L cannot possibly be a component of the resonance variety $\mathcal{R}^1(F(\mathcal{A}))$. Thus, the tangent cone formula from Theorem 5.10 is violated, and so the Milnor fiber $F(\mathcal{A})$ is not 1-formal. \diamond

In related work, Fernández de Bobadilla gave in [33] an example of a quasi-homogeneous polynomial whose Milnor fibration has trivial geometric monodromy and whose Milnor fiber is simply-connected, yet non-formal.

Zuber's example naturally leads to the following problem.

Problem 7.9. Given a rank 3 arrangement \mathcal{A} , decide whether the tangent cone formula holds for the Milnor fiber $F(\mathcal{A})$. Is this enough to guarantee that $F(\mathcal{A})$ is formal?

8 Elliptic arrangements

We conclude with another class of arrangements, this time lying in a product of elliptic curves. An especially convenient algebraic model is available for complements of ‘unimodular’ elliptic arrangements. Comparing the resonance varieties of this model to those of its cohomology algebra shows that complements of elliptic arrangements may be non-formal.

8.1 Complements of elliptic arrangements

Let $E = \mathbb{C}/\mathbb{Z}^2$ be an elliptic curve. We denote by $E^{\times n}$ be the n -fold product of such a curve. This is an abelian variety, with group law inherited from addition in \mathbb{C}^n .

An *elliptic arrangement* in $E^{\times n}$ is a finite collection of fibers of group homomorphisms from $E^{\times n}$ to E . Each “elliptic hyperplane” $H \subset E^{\times n}$ may be written as $H = f^{-1}(\zeta)$, for some point $\zeta \in E$ and some homomorphism $f: E^{\times n} \rightarrow E$ given by

$$f(z_1, \dots, z_n) = \sum_{j=1}^n c_j z_j, \quad (40)$$

where $c_j \in \mathbb{Z}$. Thus, an arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $E^{\times n}$ is determined by an integral $m \times n$ matrix $C = (c_{ij})$ and a point $\zeta = (\zeta_1, \dots, \zeta_m) \in E^{\times m}$. We will write $\text{corank}(\mathcal{A}) := n - \text{rank}(C)$ and say that \mathcal{A} is *essential* if its corank is zero.

Let $L(\mathcal{A})$ denote the collection of all connected components of intersections of zero or more elliptic hyperplanes from \mathcal{A} . Then $L(\mathcal{A})$ forms a finite poset under inclusion. We say that \mathcal{A} is *unimodular* if all subspaces in $L(\mathcal{A})$ are connected.

Now let $M(\mathcal{A}) = E^{\times n} \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of our elliptic arrangement. This space is a smooth, quasi-projective variety. Moreover, as shown in [18], the complement $M(\mathcal{A})$ has the homotopy type of a CW-complex of dimension $n + r$, where $r = \text{corank}(\mathcal{A})$. Furthermore, if $r = 0$, then $M(\mathcal{A})$ is a Stein manifold.

8.2 An algebraic model

Using the spectral sequence analyzed by Totaro in [71], Bibby constructs in [7] an algebraic model for the complement of a unimodular elliptic arrangement. (An alternative approach is given by Dupont in [25].) Let us briefly review this

construction, which generalizes the Gysin model of $E^* = E \setminus \{0\}$ described in Example 6.2.

Let a, b be the standard generators of $H^1(E, \mathbb{Z}) = \mathbb{Z}^2$. Applying the Künneth formula, we may identify the cohomology ring $H^*(E^{\times n}, \mathbb{Z})$ with the exterior algebra $\bigwedge(a_1, b_1, \dots, a_n, b_n)$. For a homomorphism $f: E^{\times n} \rightarrow E$ as in (40), the induced homomorphism in cohomology, $f^*: H^*(E, \mathbb{Z}) \rightarrow H^*(E^{\times n}, \mathbb{Z})$, is given by

$$f^*(a) = \sum_{j=1}^n c_j a_j, \quad f^*(b) = \sum_{j=1}^n c_j b_j. \quad (41)$$

Given an arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $E^{\times n}$, realize each elliptic hyperplane H_i as a coset of the kernel of a homomorphism $f_i: E^{\times n} \rightarrow E$. Next, consider the graded algebra

$$A_{\mathbb{Z}}(\mathcal{A}) = \bigwedge(a_1, b_1, \dots, a_n, b_n, e_1, \dots, e_m) / I(\mathcal{A}), \quad (42)$$

where $I(\mathcal{A})$ is the (homogeneous) ideal generated by the Orlik–Solomon relations (39) among the generators e_i , together with the elements

$$f_i^*(a)e_i, \quad f_i^*(b)e_i, \quad 1 \leq i \leq m. \quad (43)$$

Define a degree 1 differential d on $A_{\mathbb{Z}}(\mathcal{A})$ by setting $da_i = db_i = 0$ and

$$de_i = f_i^*(a) \wedge f_i^*(b), \quad (44)$$

and extending d to the whole algebra by the graded Leibniz rule. Finally, let $A(\mathcal{A}) = A_{\mathbb{Z}}(\mathcal{A}) \otimes \mathbb{C}$, and extend d to $A(\mathcal{A})$ in the obvious way.

Theorem 8.1 ([7]). *Let \mathcal{A} be a unimodular elliptic arrangement, and let $(A(\mathcal{A}), d)$ be the (rationally defined) CDGA constructed above. There is then a weak equivalence $A_{\text{PL}}(M(\mathcal{A})) \simeq A(\mathcal{A})$ preserving \mathbb{Q} -structures.*

In particular, we have an isomorphism $H^*(M(\mathcal{A}), \mathbb{C}) \cong H^*(A(\mathcal{A}), d)$. Using this result, we obtain the following form of the tangent cone theorem for elliptic arrangements (the analogue of Theorem 6.5 in this context).

Theorem 8.2. *Let \mathcal{A} be a unimodular elliptic arrangement. Then, for each $i \geq 0$,*

$$\tau_1(\mathcal{V}^i(M(\mathcal{A}))) = \text{TC}_1(\mathcal{V}^i(M(\mathcal{A}))) = \mathcal{R}^i(A(\mathcal{A})) \subseteq \mathcal{R}^i(M(\mathcal{A})), \quad (45)$$

with equality for $i \leq q$ if $M(\mathcal{A})$ is q -formal.

Proof. The CDGA model $(A(\mathcal{A}), d)$ is finite-dimensional, since the underlying graded algebra $A(\mathcal{A})$ is a quotient of a finitely-generated exterior algebra. Furthermore, this model has positive weights: simply assign weight 1 to the generators a_i, b_i and weight 2 to the generators e_i .

Using now Theorem 8.1, the rest of the argument from Theorem 6.5 goes through, once we replace the Gysin model $A(M(\mathcal{A}))$ with the Bibby model $A(\mathcal{A})$. \square

As a consequence, each resonance variety $\mathcal{R}^i(A(\mathcal{A}))$ is a union of rationally defined linear subspaces. As we shall see in §8.3, that's not always true for the resonance variety $\mathcal{R}^i(M(\mathcal{A}))$, in which case the last inclusion from (45) fails to be an equality, and $M(\mathcal{A})$ fails to be i -formal.

It is worth noting that the Orlik–Solomon-type relations for the model $A(\mathcal{A})$ are combinatorially determined, yet the relations (43) depend on the actual defining equations for the arrangement. This observation leads to the following natural question.

Problem 8.3. Let \mathcal{A} be a unimodular elliptic arrangement. Are the resonance varieties $\mathcal{R}^i(A(\mathcal{A}))$ and $\mathcal{R}^i(M(\mathcal{A}))$ determined by the intersection lattice of \mathcal{A} ? Furthermore, is there a combinatorial criterion to decide whether the two varieties coincide, and, if so, whether the complement $M(\mathcal{A})$ is formal?

8.3 Ordered configurations on an elliptic curve

The configuration space of n points on an elliptic curve, $\text{Conf}(E, n)$, is the complement of the elliptic braid arrangement, which is the arrangement in $E^{\times n}$ defined by the equations $z_i = z_j$ for $1 \leq i < j \leq n$. This space is a $K(\pi, 1)$, with $\pi = PE_n$, the elliptic pure braid group on n strings.

The resonance varieties $\mathcal{R}^1(\text{Conf}(E, n))$ were computed in [23], while the positive-dimensional components of $\mathcal{V}^1(\text{Conf}(E, n))$ were computed by Dimca in [19]. An alternate way to perform this computation is to use work of Feler, namely, [30, Theorem 3.1].

Since $E = S^1 \times S^1$ is a topological group, the space $\text{Conf}(E, n)$ splits up to homeomorphism as a direct product, $\text{Conf}(E^*, n-1) \times E$, where E^* denotes the elliptic curve E with the identity removed. Thus, for our purposes here it is enough to consider the configuration spaces $\text{Conf}(E^*, n-1)$. In the next example, we work out in detail the case when $n = 3$.

Example 8.4. Let $X = \text{Conf}(E^*, 2)$ be the configuration space of 2 labeled points on a punctured elliptic curve. This is the complement of the arrangement \mathcal{A} in $E^{\times 2}$ defined by the polynomial $f = z_1 z_2 (z_1 - z_2)$.

By Theorem 8.1, the space X admits as a model the CDGA (A, d) , where A is the exterior algebra on generators $a_1, b_1, a_2, b_2, e_1, e_2, e_3$ in degree 1, modulo the ideal generated by the quadrics

$$a_1 e_1, b_1 e_1, a_2 e_2, b_2 e_2, (a_1 - a_2) e_3, (b_1 - b_2) e_3, (e_1 - e_2)(e_1 - e_3),$$

while the differential $d: A \rightarrow A$ is given by $da_1 = db_1 = da_2 = db_2 = 0$ and

$$de_1 = a_1b_1, \quad de_2 = a_2b_2, \quad de_3 = (a_1 - a_2)(b_1 - b_2).$$

Identify $H^1(A) = \mathbb{C}^4$, with basis the classes represented by a_1, b_1, a_2, b_2 , and let $S = \mathbb{C}[x_1, y_1, x_2, y_2]$ be the corresponding polynomial ring. Fixing bases as above for $A^1 = \mathbb{C}^7$ and $\{a_1b_1, a_1a_2, a_1b_2, b_1a_2, b_1b_2, a_2b_2, a_2e_1, b_2e_1, a_1e_2, b_1e_2, a_1e_3, b_1e_3, e_1e_2, e_1e_3\}$ for $A^2 = \mathbb{C}^{14}$, we find that the boundary maps for the chain complex $A \cdot \otimes S$ are given by

$$\partial_2 = \begin{pmatrix} -y_1 & -x_2 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & -x_2 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & y_1 & 0 & -y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & y_1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_1 & y_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x_1 + x_2 & y_1 + y_2 & 0 & 0 \end{pmatrix}$$

and $\partial_1 = (x_1 \ y_1 \ x_2 \ y_2 \ 0 \ 0 \ 0)$. Computing homology, we find that $H_1(A \cdot \otimes S)$ is presented by the S -linear map $\varphi: S^7 \rightarrow S^3$ with matrix

$$\varphi = \begin{pmatrix} y_2 & x_2 & y_2 & x_2 & -y_2 & -x_2 \\ y_1 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & x_2 & y_1 & x_1 \end{pmatrix}.$$

By Theorem 2.7, the resonance variety $\mathcal{R}^1(A)$ is the zero locus of the ideal of 3×3 minors of φ . An easy computation shows that this variety is the union of three planes in \mathbb{C}^4 ,

$$\mathcal{R}^1(A) = \{x_1 = y_1 = 0\} \cup \{x_2 = y_2 = 0\} \cup \{x_1 + x_2 = y_1 + y_2 = 0\}.$$

On the other hand, the ring $H^*(A) \cong H^*(X, \mathbb{C})$ is the exterior algebra on generators a_1, a_2, b_1, b_2 in degree 1, modulo the ideal spanned by $a_1b_2 + a_2b_1$, a_1b_1 , and a_2b_2 . Proceeding as above, we see that

$$H_1(H_*(A) \otimes S) = \text{coker} \begin{pmatrix} y_2 & x_2 & -y_1 & -x_1 \\ y_1 & x_1 & 0 & 0 \\ 0 & 0 & y_2 & x_2 \end{pmatrix}.$$

Hence, the first resonance variety of X is an irreducible quadric hypersurface in \mathbb{C}^4 , given by

$$\mathcal{R}^1(X) = \{x_1y_2 - x_2y_1 = 0\}.$$

It follows from Corollary 5.11 that the configuration space $X = \text{Conf}(E^*, 2)$ is not 1-formal, a result already known from [6], [23].

Turning now to homology with coefficients in rank 1 local systems, direct computation (recorded in [65, Example 8.2]) shows that the first characteristic variety of X consists of three 2-dimensional algebraic tori inside $(\mathbb{C}^*)^4$,

$$\mathcal{V}^1(X) = \{t_1 = s_1 = 1\} \cup \{t_2 = s_2 = 1\} \cup \{t_1 t_2 = s_1 s_2 = 1\}.$$

As noted in [19, Proposition 5.1], these three subtori arise by pullback along the fibrations $\text{Conf}(E^*, 2) \rightarrow E^*$ obtained by sending a point (z_1, z_2) to z_2 , z_1 , and $z_1 z_2^{-1}$, respectively. Likewise, according to Theorem 6.6, the three planes comprising $\mathcal{R}^1(A)$ are obtained by pulling back the linear space $H^1(A(E^*)) = \mathbb{C}^2$ along the same fibrations. In particular,

$$\tau_1(\mathcal{V}^1(X)) = \text{TC}_1(\mathcal{V}^1(X)) = \mathcal{R}^1(A),$$

as predicted by Theorem 8.2.

All three varieties are 2-dimensional; thus, they are all properly contained in the 3-dimensional variety $\mathcal{R}^1(X)$. Therefore, the Tangent Cone theorem shows, once again, that X is not 1-formal. \diamond

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